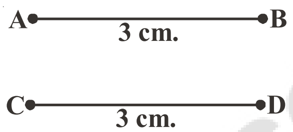
07 Triangles

7.1 Introduction

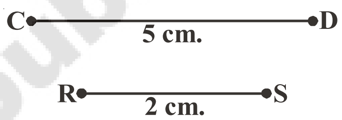
We have drawn figures with lines and curves and studied their properties. Do you remember, how to draw a line segment of a given length? All line segments are not same in size, they can be of different lengths. We also draw circles. What measure, do we need and have been used to draw a circle? It is the radius of the circle. We also draw angles equal to the given angle.

We know if the lengths of two line segments are equal then they are congruent.



$\overline{AB} \cong \overline{CD}$

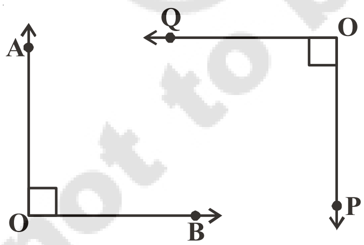
(Congruent)



$\overline{PQ} \ncong \overline{RS}$

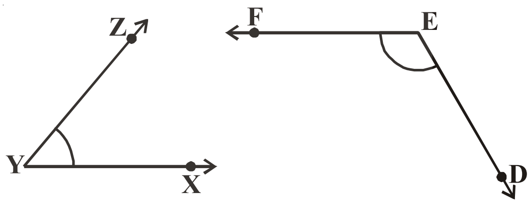
(Non-congruent)

Two angles are congruent, if their angle measure is same.



$\angle AOB \cong \angle POQ$

(Congruent)



$\angle XYZ \ncong \angle DEF$

(Non-congruent)

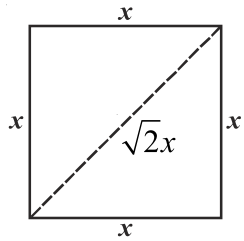
From the above examples we can say that to make or check whether the figures are same in size or not we need some specific information about the measures describing these figures.

Let's consider a square : What is the minimum information required to say whether two squares are of the same size or not?

Satya said- "I only need the measure of the side of the given squares. If the sides of given squares are equal then the squares are of identical size".

Siri said "that is right but even if the diagonals of the two squares are equal then we can say that the given squares are identical and are same in size".

Do you think both of them are right?



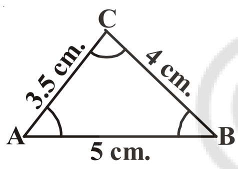
Recall the properties of a square. You can't make two different squares with sides having same measures. Can you? And the diagonals of two squares can only be equal when their sides are equal. See the given figure:

The figures that are the same in shape and size are called congruent figures ('Congruent' means equal in all respects) Hence squares that have sides with same measure are congruent and also with equal diagonals are congruent.

Note : In general, sides decide sizes and angles decide shapes.

We know if two squares are congruent and we trace one out of them on a paper and place it on other one, it will cover the other exactly.

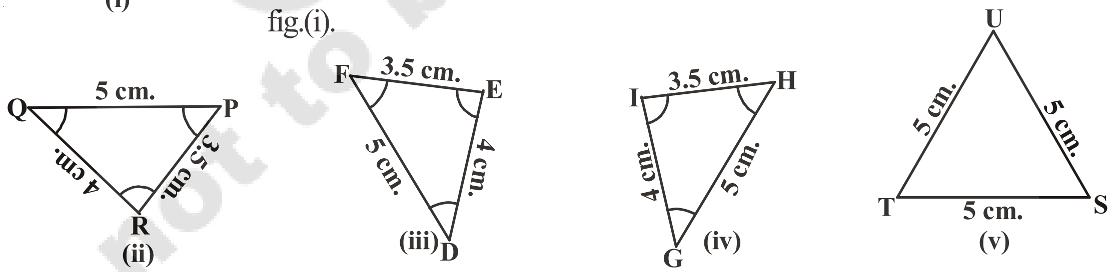
Then we can say that sides, angles, diagonals of one square are respectively equal to the sides, angles and diagonals of the other square.



(i)

Let us now consider the congruence of two triangles. We know that if two traingles are congruent then the sides and angles of one triangle are equal to the corresponding sides and angles of the other triangle.

Which of the triangles given below are congruent to triangle ABC in fig.(i).



If we trace these triangles from fig.(ii) to (v) and try to cover ΔABC. We would observe that triangles in fig.(ii), (iii) and (iv) are congruent to ΔABC while ΔTSU in fig.(v) is not congruent to ΔABC.

If ΔPQR is congruent to ΔABC, we write $\Delta PQR \cong \Delta ABC$.

Notice that when $\Delta PQR \cong \Delta ABC$, then sides of ΔPQR covers the corresponding sides of ΔABC equally and so do the angles.

That is, PQ covers AB, QR covers BC and RP covers CA; $\angle P$ covers $\angle A$, $\angle Q$ covers $\angle B$ and $\angle R$ covers $\angle C$. Also, there is a one-one correspondence between the vertices. That is, P corresponds to A, Q to B, R to C. This can be written as

$P \leftrightarrow A, Q \leftrightarrow B, R \leftrightarrow C$

Note that under order of correspondence, $\Delta PQR \cong \Delta ABC$; but it will not be correct to write $\Delta QRP \cong \Delta ABC$ as we get QR = AB, RP = BC and QP = AC which is incorrect for the given figures.

Similarly, for fig. (iii),

$FD \leftrightarrow AB, DE \leftrightarrow BC$ and $EF \leftrightarrow CA$

and $F \leftrightarrow A, D \leftrightarrow B$ and $E \leftrightarrow C$

So, $\Delta FDE \cong \Delta ABC$ but writing $\Delta DEF \cong \Delta ABC$ is not correct.

Now you give the correspondence between the triangle in fig.(iv) and ΔABC.

So, it is necessary to write the correspondence of vertices correctly for writing of congruence of triangles.

Note that **corresponding parts of congruent triangles** are **equal** and we write in short as 'CPCT' for *corresponding parts of congruent triangles*.

Do This

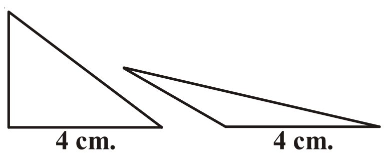
1. There are some statements given below. Write whether they are true or false :
   - i. Two circle are always congruent. ( )
   - ii. Two line segments of same length are always congruent. ( )
   - iii. Two right angle triangles are sometimes congruent. ( )
   - iv. Two equilateral triangles with their sides equal are always congruent. ( )
2. Which minimum measurements do you require to check if the given figures are congruent:
   - i. Two rectangles
   - ii. Two rhombuses

7.2 Criteria for Congruence of Triangles

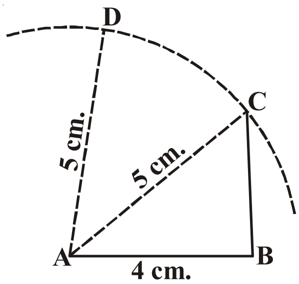
You have learnt the criteria for congruency of triangle in your earlier class.

Is it necessary to know all the three sides and three angles of a triangle to make a unique triangle?

Draw two triangles with one side 4 cm. Can you make two different triangles with one side of 4 cm? Discuss with your friends. Do you all get congruent triangles? You can draw types of triangles if one side is given say 4 cm.

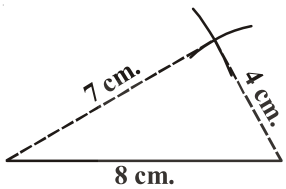


Now take two sides as 4 cm. and 5 cm. and draw as many triangles as you can. Do you get congruent triangles?



We can make different triangles even with these two given measurements.

Now draw triangles with sides 4 cm., 7 cm. and 8 cm.



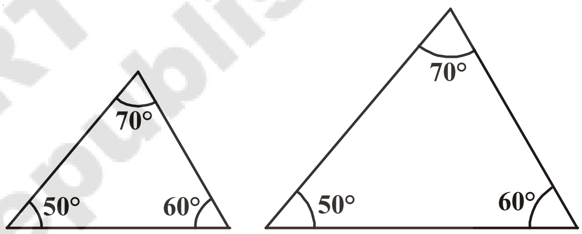
Can you draw two different triangles?

You find that with measurement of these three sides, we can make a unique triangle. If at all you draw the triangles with these dimensions they will be congruent to this unique triangle.

Now take three angles of your choice, of course The sum of the angles must be 180°. Draw two triangles for your choosen angle measurement.

Mahima finds that she can make different triangles by using three angle measurement.

$\angle A = 50°$, $\angle B = 70°$, $\angle C = 60°$



So it seems that knowing the 3 angles in not enough to make a specific triangle.

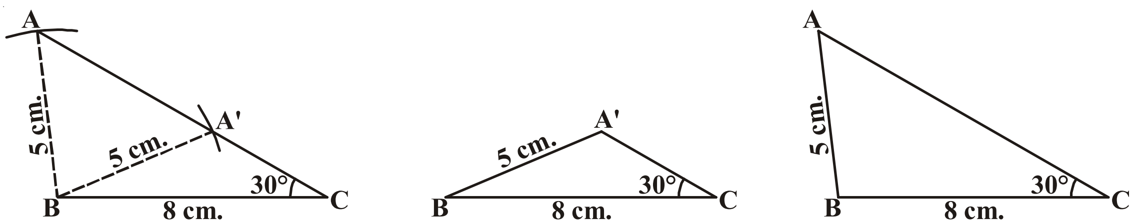
Sharif thought that if two angles are given then he could easily find the third one by using the property of sum of the angles is triangle. So measures of two angles is enough to draw the triangle but not uniquely. Hence giving 3 or 2 angles is not adequate. We need at least three specific and independent measurements (elements) to make a unique triangle.

Now try to draw two distinct triangles with each sets of these three measurements:

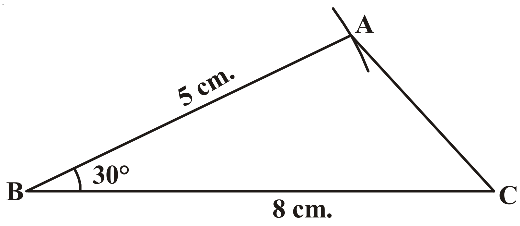
i. ΔABC where AB = 5 cm., BC = 8 cm., $\angle C = 30°$

ii. ΔABC where AB = 5 cm., BC = 8 cm., $\angle B = 30°$

(i) Are you able to draw a unique triangle with the given measurements, draw and check with your friends.



Here we can draw two different triangles ΔABC and $\Delta A'BC$ with given measurements. Now draw two triangles with given measurements (ii). What do you observe? They are congruent triangles. Aren't they?



In the other words you can draw a unique triangle with the measurements given in case(ii).

Have you noticed the order of measures given in case (i) & case (ii) ? In case (i) two sides and one angle are given which is not an included angle but in case (ii) included angle is given along with two sides. Thus given two sides and one angle i.e. three independent measures is not the only criteria to make a unique triangle. But the order of given measurements to construct a triangle also plays a vital role in making a unique triangle.

7.3 CONGRUENCE OF TRIANGLES

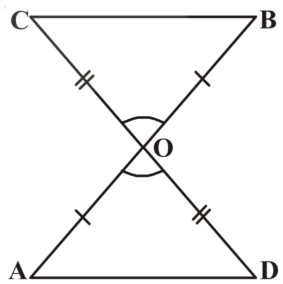
The above has an implication for checking the congruency of triangles. If we have two triangles with one side equal or two triangles with all 3 angles equal, we can not conclude that triangles are congruent as there are more than one triangle possible with these specifications. Even when we have two sides and an angle equal we cannot say that the triangles are congruent unless the angle is between the given sides. We can say that the SAS (side angle side) congruency rule holds but not SSA or ASS.

We take this as the first criterion for congruency of triangles and prove the other criteria through this.

Axiom (SAS congruence rule): *Two triangles are congruent if two sides and the included angle of one triangle are equal to the two sides and the included angle of the other triangle.*

Example-1. In the given Figure AB and CD are intersecting at 'O', OA = OB and OD = OC. Show that

(i) $\Delta AOD \cong \Delta BOC$ and (ii) AD || BC.



Solution : (i) you may observe that in ΔAOD and ΔBOC,

OA = OB (given)

OD = OC (given)

Also, since $\angle AOD$ and $\angle BOC$ form a pair of vertically opposite angles, we have

$\angle AOD = \angle BOC$.

So, $\Delta AOD \cong \Delta BOC$ (by the SAS congruence rule)

(ii) In congruent triangles AOD and BOC, the other corresponding parts are also equal.

So, $\angle OAD = \angle OBC$ and these form a pair of alternate angles for line segments AD and BC.

Therefore AD || BC

Example-2. AB is a line segment and line l is its perpendicular bisector. If a point P lies on l, show that P is equidistant from A and B.

Solution : Line $l \perp AB$ and passes through C which is the mid-point of AB

We have to show that PA = PB.

Consider ΔPCA and ΔPCB.

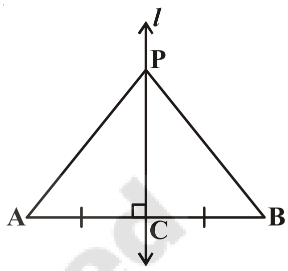
We have AC = BC (C is the mid-point of AB)

∠PCA = ∠PCB = 90° (Given)

PC = PC (Common)

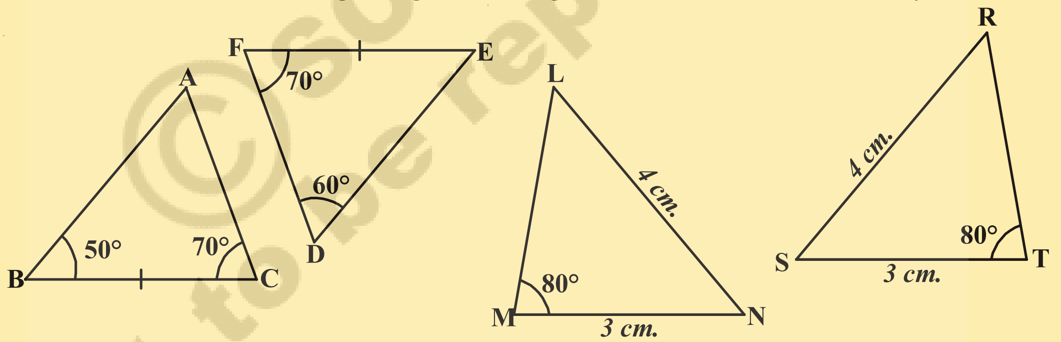
So, ΔPCA ≅ ΔPCB (SAS rule)

and so, PA = PB, as they are corresponding sides of congruent triangles.



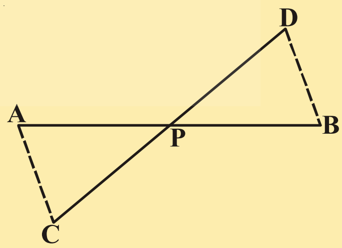
DO THESE

1. State whether the following triangles are congruent or not? Give reasons for your answer.



2. In the given figure, the point P bisects AB and DC. Prove that

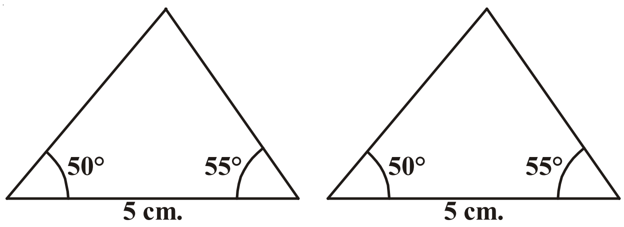
ΔAPC ≅ ΔBPD



7.3.1 Other Congruence Rules

Try to construct two triangles in which two of the angles are 50° and 55° and the side on which both these angles lie being 5cm.

Cut out these triangles and place one on the other. What do you observe? You will find that both the triangles are congruent. This result is the angle-side-angle criterion for congruence and is written as ASA criterion you have seen this in earlier classes. Now let us state and prove the result. Since this result can be proved, it is called a theorem and to prove it, we use the SAS axiom for congruence.



Theorem 7.1 (ASA congruence rule) : Two triangles are congruent, if two angles and the included side of one triangle are equal to two angles and the included side of the other triangle.

Given: In ΔABC and ΔDEF

$$\angle B = \angle E,\ \angle C = \angle F \text{ and } \overline{BC} = \overline{EF}$$

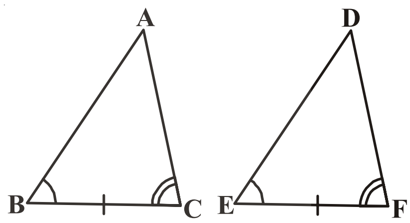
Required To Prove (RTP): ΔABC ≅ Δ DEF

Proof: There will be three posibilities. The possiblities between $\overline{AB}$ and $\overline{DE}$ are either $\overline{AB} > \overline{DE}$ or $\overline{DE} > \overline{AB}$ or $\overline{DE} = \overline{AB}$.

We will consider all these cases and see what does it mean for ΔABC and ΔDEF.

Case (i): Let $\overline{AB} = \overline{DE}$ Now what do we observe?

Consider ΔABC and ΔDEF



$\overline{AB} = \overline{DE}$ (Assumed)

$\angle B = \angle E$ (Given)

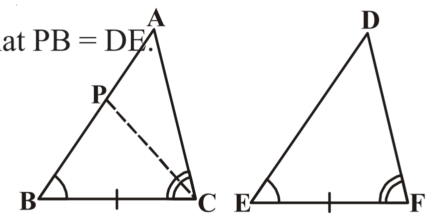
$\overline{BC} = \overline{EF}$ (Given)

So, ΔABC ≅ ΔDEF (By SAS congruency axiom)

Case (ii): The second possibility is AB > DE.

So, we can take a point P on AB such that PB = DE.

Now consider ΔPBC and ΔDEF



$\overline{PB} = \overline{DE}$ (by construction)

$\angle B = \angle E$ (given)

$\overline{BC} = \overline{EF}$ (given)

So, ΔPBC ≅ ΔDEF (by SAS congruency axiom)

Since the triangles are congruent their corresponding parts will be equal

So, $\angle PCB = \angle DFE$

But, $\angle ACB = \angle DFE$ (given)

So $\angle ACB = \angle PCB$ (from the above)

Is this possible?

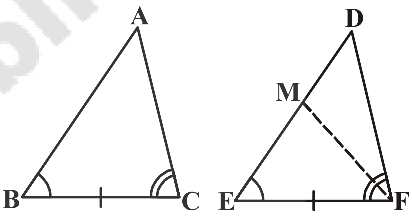
This is possible only if P coincides with A

(or) $\overline{BA} = \overline{ED}$

So, $\Delta ABC \cong \Delta DEF$ (By SAS congruency axiom)

(**Note :** We have shown above that if $\angle B = \angle E$ and $\angle C = \angle F$ and $\overline{BC} = \overline{EF}$ then $\overline{AB} = \overline{DE}$ and the two triangles are congruency by SAS rule).

Case (iii): The third possibility is $\overline{AB} < \overline{DE}$



We can choose a point M on DE such that ME = AB and repeating the arguments as given in case (ii), we can conclude that $\overline{AB} = \overline{DE}$ and so, $\Delta ABC \cong \Delta DEF$. Look at the figure and try to prove it yourself.

Suppose, now in two triangles two pairs of angles and one pair of corresponding sides are equal but the side is not included between the corresponding equal pairs of angles. Are the triangles still congruent? You will observe that they are congruent. Can you reason out why?

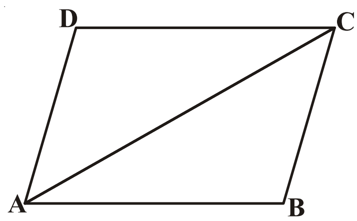
You know that the sum of the three angles of a triangle is 180°. So if two pairs of angles are equal, the third pair is also equal (180° – sum of equal angles).

So, two *triangles are congruent if any two pairs of angles and one pair of corresponding sides are equal.* We may call it as the **AAS Congruence Rule.** Let us now take some more examples.

Example-3. In the given figure, AB|| DC and AD|| BC

Show that $\Delta ABC \cong \Delta CDA$

Solution : Consider ΔABC and ΔCDA



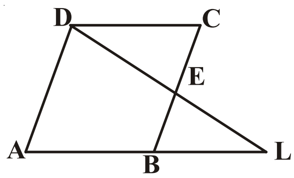
$\angle BAC = \angle DCA$ (alternate interior angles)

AC = CA (common side)

$\angle BCA = \angle DAC$ (alternate interior angles)

$\Delta ABC \cong \Delta CDA$(by ASA congruency)

Example-4. In the given figure, AL || DC, E is mid point of BC. Show that $\Delta EBL \cong \Delta ECD$



Solution : Consider ΔEBL and ΔECD

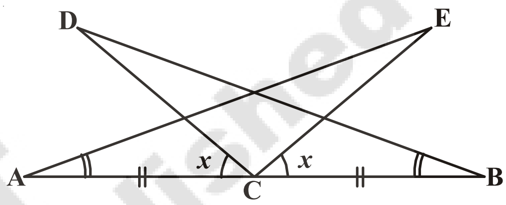
$\angle BEL = \angle CED$ (vertically opposite angles)

BE = CE (since E is mid point of BC)

$\angle EBL = \angle ECD$ (alternate interior angles)

$\Delta EBL \cong \Delta ECD$ (by ASA congruency)

Example-5. Use the information given in the adjoining figure, to prove :



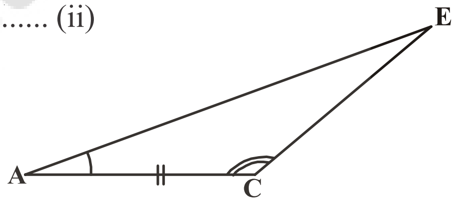
(i) $\Delta DBC \cong \Delta EAC$

(ii) DC = EC.

Solution : Let $\angle ACD = \angle BCE = x$

$\therefore \angle ACE = \angle DCE + \angle ACD = \angle DCE + x$ ...... (i)

$\therefore \angle BCD = \angle DCE + \angle BCE = \angle DCE + x$ ...... (ii)



From (i) and (ii), we get : $\angle ACE = \angle BCD$

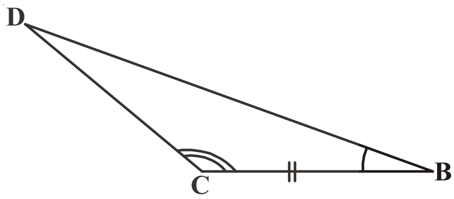
Now in ΔDBC and ΔEAC,

$\angle ACE = \angle BCD$ (proved above)

BC = AC [Given]

$\angle CBD = \angle EAC$ [Given]

$\Delta DBC \cong \Delta EAC$ [By A.S.A]

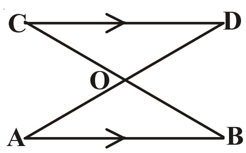


since $\Delta DBC \cong \Delta EAC$

DC = EC.(by CPCT)

Example-6. Line-segment AB is parallel to another line-segment CD. O is the mid-point of AD.

Show that (i) $\Delta AOB \cong \Delta DOC$ (ii) O is also the mid-point of BC.



Solution : (i) Consider ΔAOB and ΔDOC.

$\angle ABO = \angle DCO$ (Alternate angles as AB || CD and BC is the transversal)

$\angle AOB = \angle DOC$ (Vertically opposite angles)

OA = OD (Given)

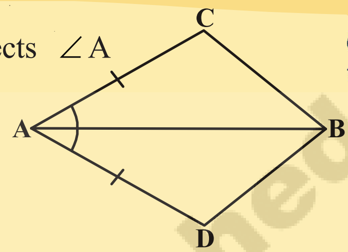
Therefore, ΔAOB ≅ ΔDOC (AAS rule)

(ii) OB = OC (CPCT)

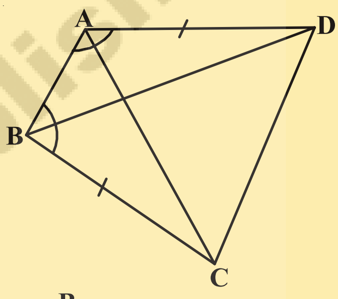
So, O is the mid-point of BC.

EXERCISE - 7.1

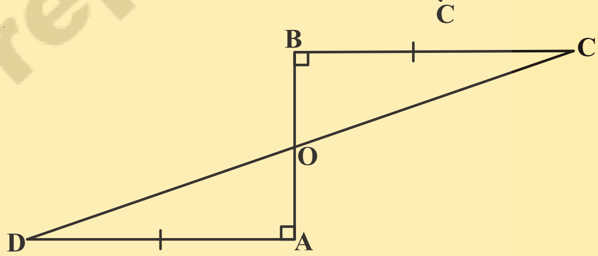
1. In quadrilateral ACBD, AC = AD and AB bisects ∠A Show that ΔABC ≅ ΔABD. What can you say about BC and BD?



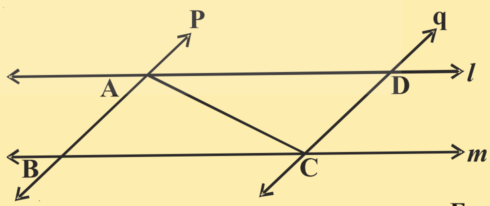
2. ABCD is a quadrilateral in which AD = BC and ∠DAB = ∠CBA Prove that
   (i) ΔABD ≅ ΔBAC
   (ii) BD = AC
   (iii) ∠ABD = ∠BAC



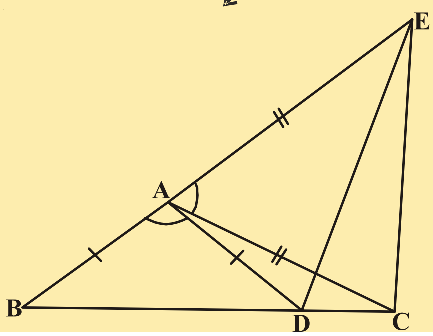
3. AD and BC are equal and perpendiculars to a line segment AB. Show that CD bisects AB.



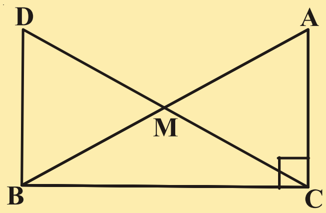
4. *l* and *m* are two parallel lines intersected by another pair of parallel lines p and q . Show that ΔABC ≅ ΔCDA



5. In the adjacent figure, AC = AE, AB = AD and ∠BAD = ∠EAC. Show that BC = DE.



6. In right triangle ABC, right angle is at C, M is the mid-point of hypotenuse AB. C is joined to M and produced to a point D such that DM = CM. Point D is joined to point B (see figure). Show that :



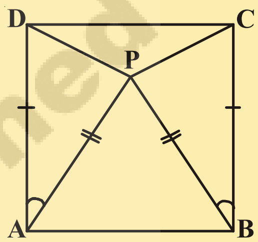
(i) $\Delta AMC \cong \Delta BMD$

(ii) $\angle DBC$ is a right angle

(iii) $\Delta DBC \cong \Delta ACB$

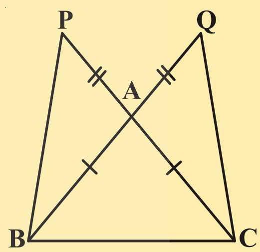
(iv) $CM = \frac{1}{2} AB$

7. In the adjacent figure ABCD is a square and ΔAPB is an equilateral triangle. Prove that $\Delta APD \cong \Delta BPC$.

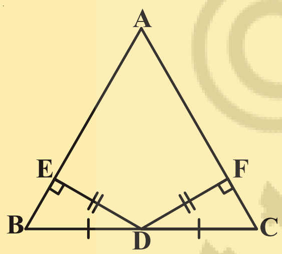


(**Hint :** In ΔAPD and ΔBPC $\overline{AD} = \overline{BC}$, $\overline{AP} = \overline{BP}$ and $\angle PAD = \angle PBC = 90^o - 60^o = 30^o$]

8. In the adjacent figure ΔABC is isosceles as $\overline{AB} = \overline{AC}$, $\overline{BA}$ and $\overline{CA}$ are produced to Q and P such that $\overline{AQ} = \overline{AP}$. Show that $\overline{PB} = \overline{QC}$



(**Hint :** Compare ΔAPB and ΔACQ)



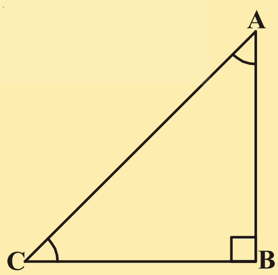
9. In the adjacent figure ΔABC, D is the midpoint of BC. $DE \perp AB$, $DF \perp AC$ and DE = DF. Show that $\Delta BED \cong \Delta CFD$.

10. If the bisector of an angle of a triangle also bisects the opposite side, prove that the triangle is isosceles.

11. In the given figure ABC is a right triangle and right angled at B such that $\angle BCA = 2\angle BAC$.

Show that hypotenuse AC = 2BC.

(**Hint** : Produce CB to a point D that BC = BD)

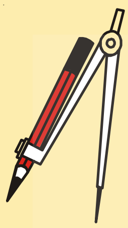


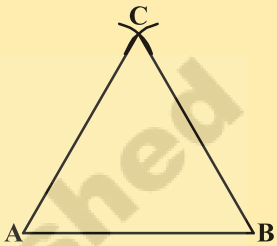
7.4 SOME PROPERTIES OF A TRIANGLE

In the above section you have studied two criteria for the congruence of triangles. Let us now apply these results to study some properties related to a triangle whose two sides are equal.

ACTIVITY

i. To contruct a triangle using compass, take any measurement and draw a line segment AB. Now open a compass with sufficient length and put it on point A and B and draw an arc. Which type of triangle do you get? Yes this is an isosceles triangle. So, ΔABC in figure is an isosceles triangle with AC = BC. Now measure ∠A and ∠B. What do you observe?





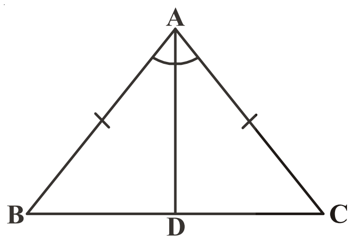
ii. Cut some isosceles triangles.

Now fold the triangle so that two congruent sides fit precisely one on top of the other. What do you notice about ∠A and ∠B?

You may observe that in each such triangle, the angles opposite to the equal sides are equal. This is a very important result and is indeed true for any isosceles triangle. It can be proved as shown below.

Theorem-7.2 : Angles opposite to equal sides of an isosceles triangle are equal.

This result can be proved in many ways. One of the proofs is given here.



Given: ΔABC is an isosceles triangle in which AB = AC.

RTP: ∠B = ∠C.

Construction: Let us draw the bisector of ∠A and let D be the point of intersection of this bisector of ∠A and BC.

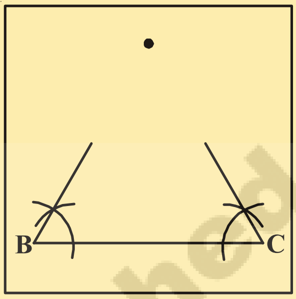
Proof : In ΔBAD and ΔCAD,

AB = AC	(Given)
∠BAD = ∠CAD	(By construction)
AD = AD	(Common)
So, ΔBAD ≅ ΔCAD	(By SAS congruency axiom)
So, ∠ABD = ∠ACD	(By CPCT)
i.e., ∠B = ∠C	(Same angles)

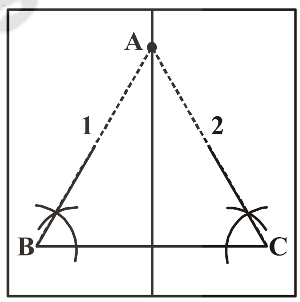
Is the converse also true? That is "If two angles of any triangle are equal, can we conclude that the sides opposite to them are also equal?"

ACTIVITY

1. On a tracing paper draw a line segment BC of length 6cm.
2. From vertices B and C draw rays with angle 60^0 each. Name the point A where they meet.
3. Fold the paper so that B and C fit precisely on top of each other. What do you observe? Is AB = AC?



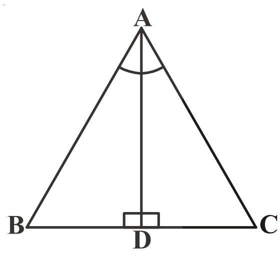
Repeat this activity by taking different angles for ∠B and ∠C. Each time you will observe that the sides opposite to equal angles are equal. So we have the following



Theorem-7.3 : The sides opposite to equal angles of a triangle are equal.

This is the converse of previous Theorem. The student is advised to prove this using ASA congruence rule.

Example-7. In ΔABC, the bisector AD of A is perpendicular to side BC Show that AB = AC and ΔABC is isosceles.



Solution : In ΔABD and ΔACD,

∠BAD = ∠CAD (Given)

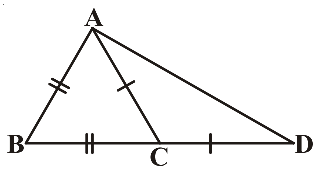
AD = AD (Common side)

∠ADB = ∠ADC = 90° (Given)

So, ΔABD ≅ ΔACD (ASA rule)

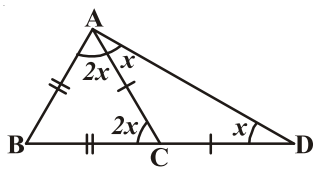
So, AB = AC (CPCT)

or, ΔABC is an isosceles triangle.



Example-8. In the adjacent figure, AB = BC and AC = CD.

Prove that : ∠BAD : ∠ADB = 3 : 1.



Solution : Let ∠ADB = x

In ΔACD, $AC = CD$

$\Rightarrow \quad \angle CAD = \angle CDA = x$

and, ext. $\angle ACB = \angle CAD + \angle CDA$

$= x + x = 2x$

$\Rightarrow \quad \angle BAC = \angle ACB = 2x$. ($\because$ In ABC, AB = BC)

$\therefore \quad \angle BAD = \angle BAC + \angle CAD$

$= 2x + x = 3x$

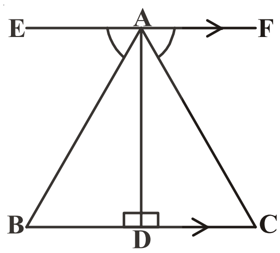
And, $\dfrac{\angle BAD}{\angle ADB} = \dfrac{3x}{x} = \dfrac{3}{1}$

i.e., $\angle BAD : \angle ADB = 3 : 1$.

Hence Proved.

Example-9. In the given figure, AD is perpendicular to BC and EF || BC, if $\angle EAB = \angle FAC$, show that triangles ABD and ACD are congruent.

Also, find the values of x and y if $AB = 2x + 3$, $AC = 3y + 1$, $BD = x$ and $DC = y + 1$.



Solution : AD is perpendicular to EF

$\Rightarrow \quad \angle EAD = \angle FAD = 90^0$

$\angle EAB = \angle FAC$ (given)

$\Rightarrow \quad \angle EAD - \angle EAB = \angle FAD - \angle FAC$

$\Rightarrow \quad \angle BAD = \angle CAD$

In ΔABD and ΔACD

$\angle BAD = \angle CAD$ [proved above]

$\angle ADB = \angle ADC = 90^0$ [Given AD is perpendicular on BC]

and $AD = AD$

$\therefore \quad \Delta ABD \cong \Delta ACD$ [By ASA]

Hence proved.

$\angle ABD = \angle ACD \Rightarrow AB = AC$ and $BD = CD$ [By C.P.C.T]

$\Rightarrow 2x + 3 = 3y + 1$ and $x = y + 1$

$\Rightarrow 2x - 3y = -2$ and $x - y = 1$

Substituting $x = 1 + y$: $2(1+y) - 3y = -2$

$2 + 2y - 3y = -2$

$-y = -2 - 2$

$-y = -4$

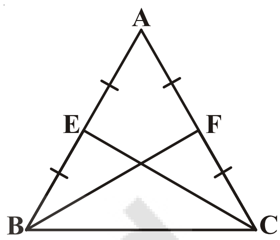
Substituting $y = 4$ in $x = 1 + y$

$x = 1 + 4$

$x = 5$

Example-10. E and F are respectively the mid-points of equal sides AB and AC of ΔABC (see figure)

Show that BF = CE



Solution : In ΔABF and ΔACE,

AB = AC (Given)

$\angle A = \angle A$ (common angle)

AF = AE (Halves of equal sides)

So, $\Delta ABF \cong \Delta ACE$ (SAS rule)

Therefore, BF = CE (CPCT)

Example-11. In an isosceles triangle ABC with AB = AC, D and E are points on BC such that BE = CD (see figure) Show that AD = AE,

Solution : In ΔABD and ΔACE,

AB = AC (given) .... (1)

$\angle B = \angle C$ (Angles opposite to equal sides) .... (2)

Also, BE = CD

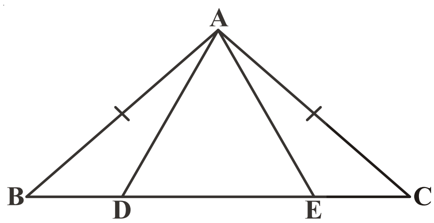
So, BE – DE = CD – DE

That is, BD = CE (3)

So, $\Delta ABD \cong \Delta ACE$

(Using (1), (2), (3) and SAS rule).

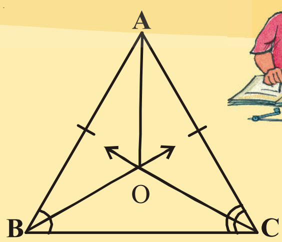
This gives AD = AE (CPCT)



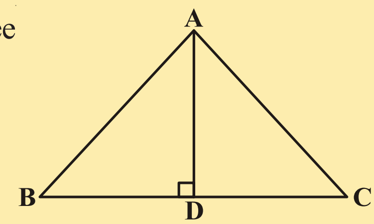
EXERCISE - 7.2

1. In an isosceles triangle ABC, with AB = AC, the bisectors of $\angle B$ and $\angle C$ intersect each other at O. Join A to O. Show that :

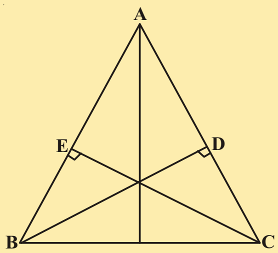
   (i) OB = OC (ii) AO bisects $\angle A$

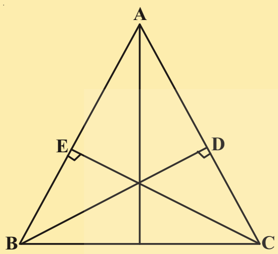


2. In ΔABC, AD is the perpendicular bisector of BC (See adjacent figure). Show that ΔABC is an isosceles triangle in which AB = AC.



3. ABC is an isosceles triangle in which altitudes BD and CE are drawn to equal sides AC and AB respectively (see figure) Show that these altitudes are equal.



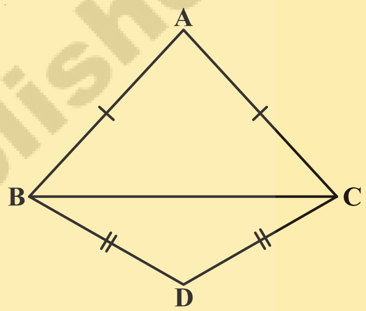


4. ABC is a triangle in which altitudes BD and CE to sides AC and AB are equal (see figure) . Show that

   (i) $\Delta ABD \cong \Delta ACE$

   (ii) AB = AC i.e., ABC is an isosceles triangle.

5. ΔABC and ΔDBC are two isosceles triangles on the same base BC (see figure). Show that $\angle ABD = \angle ACD$.



7.5 Some more criteria for congruency of triangles

Theorem 7.4 (SSS congruence rule) : Through construction we have seen that SSS congruency rule hold. This theorem can be proved using a suitable construction.

In two triangles, if the three sides of one triangle are respectively equal to the corresponding three sides of another triangle, three the two triangles are congruent.

• **Proof for SSS Congruence Rule**

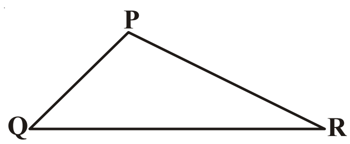
Given: ΔPQR and ΔXYZ are such that PQ = XY, QR = YZ and PR = XZ

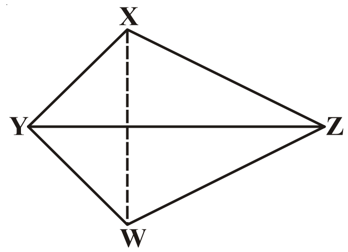
To Prove : $\Delta PQR \cong \Delta XYZ$

Construction : Draw YW such that $\angle ZYW = \angle PQR$ and WY = PQ. Join XW and WZ

Proof: In ΔPQR and ΔWYZ

QR = YZ	(Given)
$\angle PQR = \angle ZYW$	(Construction)
PQ = YW	(Construction)
$\therefore \Delta PQR \cong \Delta WYZ$	(SAS congruence axiom)





$\Rightarrow \angle P = \angle W$ and PR = WZ (CPCT)

PQ = XY (given) and PQ = YW (Construction)

$\therefore$ XY = YW

Similarly, XZ = WZ

In ΔXYW, XY = YW

$\Rightarrow \angle YWX = \angle YXW$ (In a triangle, equal sides have equal angles opposite to them)

Similarly, $\angle ZWX = \angle ZXW$

$\therefore \angle YWX + \angle ZWX = \angle YXW + \angle ZXW$

$\Rightarrow \angle W = \angle X$

Now, $\angle W = \angle P$

$\therefore \angle P = \angle X$

In ΔPQR and ΔXYZ

PQ = XY

$\angle P = \angle X$

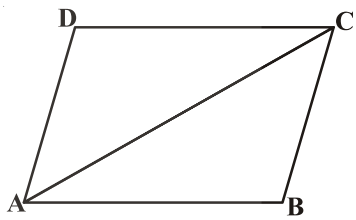
PR = XZ

$\therefore \Delta PQR \cong \Delta XYZ$ (SAS congruence criterion)

Let us see the following example based on it.

Example-12. In quadrilateral ABCD, AB = CD, BC=AD show that $\Delta ABC \cong \Delta CDA$

Consider ΔABC and ΔCDA



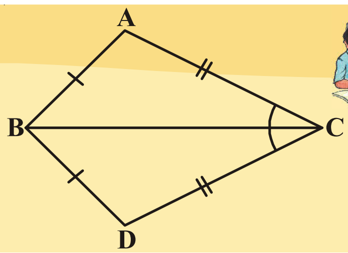
AB = CD (given)

AD = BC (given)

AC = CA (common side)

$\Delta ABC \cong \Delta CDA$ (by SSS congruency rule)

DO THIS



1. In the adjacent figure ΔABC and ΔDBC are two triangles such that $\overline{AB} = \overline{BD}$ and $\overline{AC} = \overline{CD}$. Show that $\Delta ABC \cong \Delta DBC$.

You have already seen that in the SAS congruency axiom, the pair of equal angles has to be the included angle between the pairs of corresponding equal sides and if not so, two triangles may not be congruent.

ACTIVITY

Construct a right angled triangle with hypotenuse 5 cm. and one side 3 cm. long. How many different triangles can be constructed? Compare your triangle with those of the other members of your class. Are the triangles congruent? Cut them out and place one triangle over the other with equal side placed on each other. Turn the triangle if necessary what do you observe? You will find that two right triangles are congruent, if side and hypotenous of one triangle are respectively equal to the corresponding side and hypotenous of other triangle.

Note that the right angle is not the included angle in this case. So we arrive at the following congruency rule.

Theorem 7.5 (RHS congruence rule) : If in two right triangles the hypotenuse and one side of one triangle are equal to the hypotenuse and one side of the another triangle, then the two triangles are congruent.

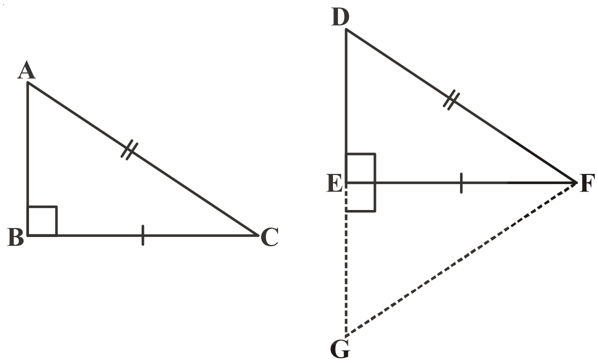
Note that RHS stands for right angle - hypotenuse-side.

Let us prove it.

Given: Two right triangles, ΔABC and ΔDEF
in which $\angle B = 90^o$ and
$\angle E = 90^o$ $AC = DF$
and $BC = EF$.

To prove: $\Delta ABC \cong \Delta DEF$

Construction: Produce DE to G
So that EG = AB. Join GF.



Proof:

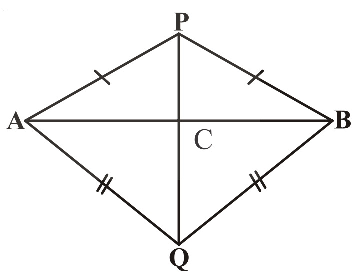
In ΔABC and ΔGEF, we have

$AB = GE$	(By construction)
$\angle B = \angle FEG$	(Each angle is a right angle (90^0))
$BC = EF$	(Given)
$\Delta ABC \cong \Delta GEF$	(By SAS criterion of congruence)
So $\angle A = \angle G$ ... (1)	(CPCT)

AC = GF … (2)	(CPCT)
Further, AC=GF and AC=DF	(From (2) and Given)
Therefore DF = GF	(From the above)
So, $\angle D = \angle G$ … (3)	(Angles opposite to equal sides are equal)
we get $\angle A = \angle D$ … (4)	(From (1) and (3))
Thus, in ΔABC and ΔDEF $\angle A = \angle D$,	(From (4))
$\angle B = \angle E$	(Given)
So, $\angle A + \angle B = \angle D + \angle E$	(on adding)
But $\angle A + \angle B + \angle C = 180^0$ and	(angle sum property of triangle)
$\angle D + \angle E + \angle F = 180^0$	
180 - $\angle C$ = 180 - $\angle F$	($\angle A + \angle B = 180^0 - \angle C$ and $\angle D + \angle E = 180^0 - \angle F$)
So, $\angle C = \angle F$, … (5)	(Cancellation laws)
Now, in ΔABC and ΔDEF, we have	
BC = EF	(given)
$\angle C = \angle F$	(from (5))
AC = DF	(given)
$\Delta ABC \cong \Delta DEF$	(by SAS axiom of congruence)

Example-13. AB is a line - segment. P and Q are points on either side of AB such that each of them is equidistant from the points A and B (See Fig). Show that the line PQ is the perpendicular bisector of AB.

Solution : You are given PA = PB and QA = QB and you have to show that PQ is perpendicular on AB and PQ bisects AB. Let PQ intersect AB at C.



Can you think of two congruent triangles in this figure ?

Let us take ΔPAQ and ΔPBQ.

In these triangles,

AP = BP (Given)

AQ = BQ (Given)

PQ = PQ (Common side)

So, $\Delta PAQ \cong \Delta PBQ$ (SSS rule)

Therefore, $\angle APQ = \angle BPQ$ (CPCT).

Now let us consider ΔPAC and ΔPBC.

You have : AP = BP (Given)

$\angle APC = \angle BPC$ ($\angle APQ = \angle BPQ$ proved above)

PC = PC (Common side)

So, $\Delta PAC \cong \Delta PBC$ (SAS rule)

Therefore, AC = BC (CPCT) .... (1)

and $\angle ACP = \angle BCP$ (CPCT)

Also, $\angle ACP + \angle BCP = 180°$ (Linear pair)

So, $2\angle ACP = 180°$

or, $\angle ACP = 90°$ .... (2)

From (1) and (2), you can easily conclude that PQ is the perpendicular bisector of AB.

[Note that, without showing the congruence of ΔPAQ and ΔPBQ, you cannot show that $\Delta PAC \cong \Delta PBC$ even though AP = BP (Given)

PC = PC (Common side)

and $\angle PAC = \angle PBC$ (Angles opposite to equal sides in ΔAPB)

It is because these results give us SSA rule which is not always valid or true for congruence of triangles as the given angle is not included between the equal pairs of sides.]

Let us take some more examples.

Example-14. P is a point equidistant from two lines *l* and *m* intersecting at point A (see figure). Show that the line AP bisects the angle between them.

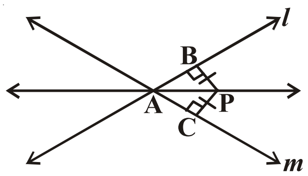
Solution : You are given that lines *l* and *m* intersect each other at A.

Let PB is perpendicular on *l* and

PC ⊥ *m*. It is given that PB = PC.

You need to show that $\angle PAB = \angle PAC$.

Let us consider ΔPAB and ΔPAC. In these two triangles,



PB = PC (Given)

$\angle PBA = \angle PCA = 90°$ (Given)

PA = PA (Common side)

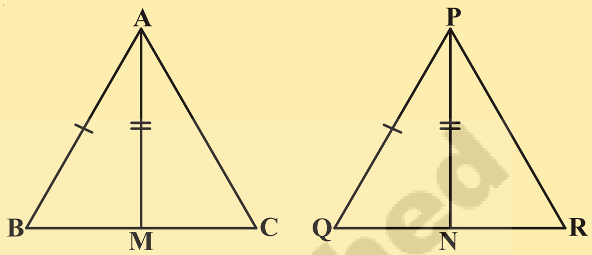
So, $\Delta PAB \cong \Delta PAC$ (RHS rule)

So, $\angle PAB = \angle PAC$ (CPCT)

Exercise - 7.3

1. AD is an altitude of an isosceles triangle ABC in which AB = AC. Show that, (i) AD bisects BC (ii) AD bisects $\angle A$.

2. Two sides AB, BC and median AM of one triangle ABC are respectively equal to sides PQ and QR and median PN of ΔPQR (See figure). Show that:



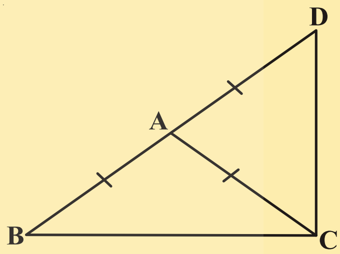
   (i) $\Delta ABM \cong \Delta PQN$

   (ii) $\Delta ABC \cong \Delta PQR$

3. BE and CF are two equal altitudes of a triangle ABC. Using RHS congruence rule, prove that the triangle ABC is isosceles.

4. ΔABC is an isosceles triangle in which AB = AC. Show that $\angle B = \angle C$. (Hint : Draw AP $\perp$ BC) (Using RHS congruence rule)

5. ΔABC is an isosceles triangle in which AB = AC. Side BA is produced to D such that AD = AB (see figure). Show that $\angle BCD$ is a right angle.

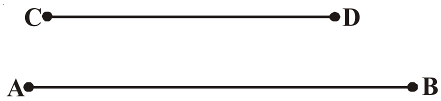


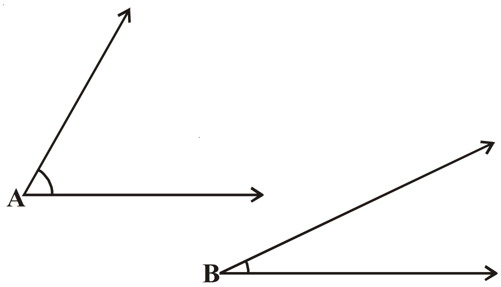
6. ABC is a right angled triangle in which $\angle A = 90^0$ and AB = AC. Show that $\angle B = \angle C$.

7. Show that the angles of an equilateral triangle are 60^0 each.

7.6 Inequalities in a Triangle

So far, you have been studying the equality of sides and angles of a triangle or triangles. Sometimes, we do come across unequal figures and we need to compare them. For example, line segment AB is greater in length as compared to line segment CD in figure (i) and $\angle A$ is greater than $\angle B$ in following figure (ii).



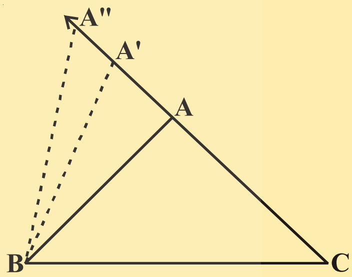


(i) (ii)

Let us now examine whether there is any relation between unequal sides and unequal angles of a triangle. For this, let us perform the following activity:

ACTIVITY

1. Draw a triangle ABC mark a point A' on CA produced (new position of it)



So, $A'C > AC$ (Comparing the lengths)

Join A' to B and complete the triangle $A'BC$.

What can you say about $\angle A'BC$ and $\angle ABC$?

Compare them. What do you observe?

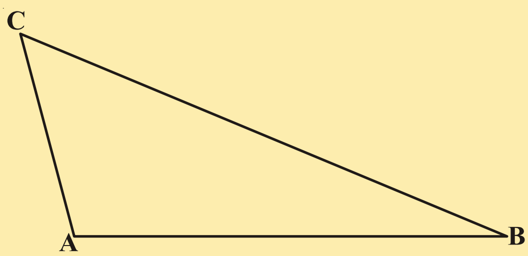
Clearly, $\angle A'BC > \angle ABC$

Continue to mark more points on CA (extended) and draw the triangles with the side BC and the points marked.

You will observe that as the length of the side AC is increases (by taking different positions of A), the angle opposite to it, that is, $\angle B$ also increases.

Let us now perform another activity-

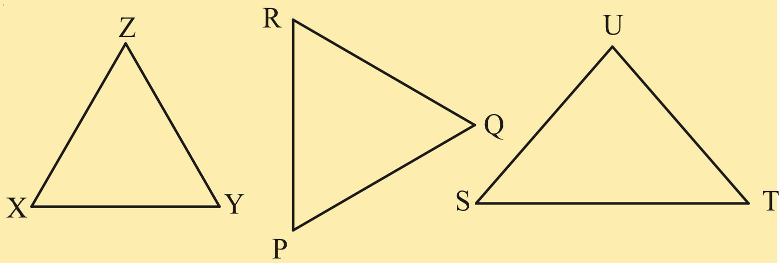
2. Construct a scalene triangle ABC (that is a triangle in which all sides are of different lengths). Measure the lengths of the sides.



Now, measure the angles. What do you observe?

In ΔABC Figure, BC is the longest side and AC is the shortest side.

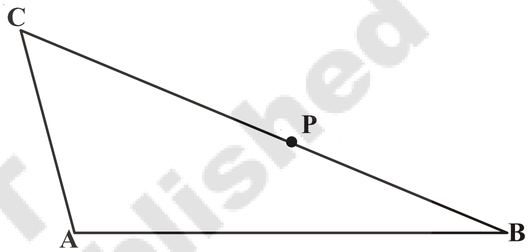
Also, ∠A is the largest and ∠B is the smallest.



Measure angles and sides of each of the above triangles, what is the relation between a side and its opposite angle when campared with another pair?

Theorem-7.6 : If two sides of a triangle are unequal, the angle opposite to the longer side is larger (or greater).

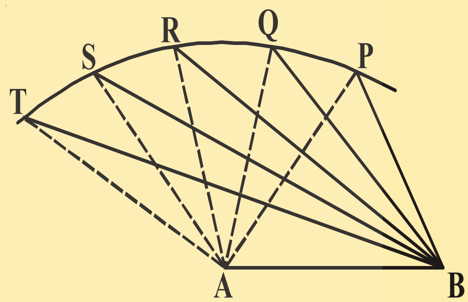
You may prove this theorem by taking a point P on BC such that CA = CP as shown in adjacent figure.



Now, let us do another activity:

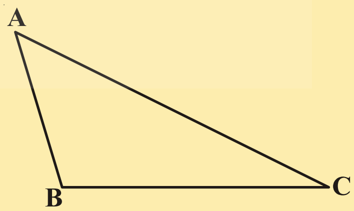
ACTIVITY

Draw a line-segment AB. With A as centre and some radius, draw an arc and mark different points say P, Q, R, S, T on it.



Join each of these points with A as well as with B (see figure). Observe that as we move from P to T, ∠A is becoming larger and larger. What is happening to the length of the side opposite to it? Observe that the length of the side is also increasing; that is ∠TAB > ∠SAB > ∠RAB > ∠QAB > ∠PAB and TB > SB > RB > QB > PB.

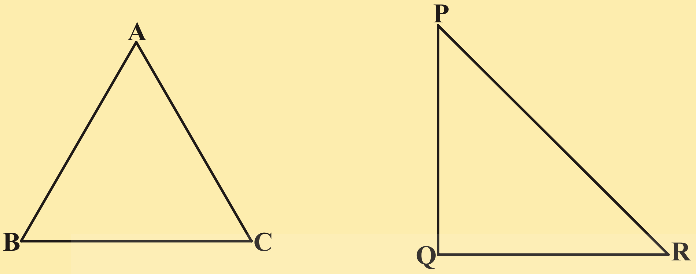
Now, draw any triangle with all angles unequal to each other. Measure the lengths of the sides (see figure).



Observe that the side opposite to the largest angle is the longest. In figure, ∠B is the largest angle and AC is the longest side.

Repeat this activity for some more triangles and we see that the converse of the above Theorem is also true.

Measure angles and sides of each triangle given below. What relation you can visualize for a side and its opposite angle in each triangle.



In this way, we arrive at the following theorem.

Theorem -7.7 : In any triangle, the side opposite to the larger (greater) angle is longer.

This theorem can be proved by the method of contradiction.

Do This

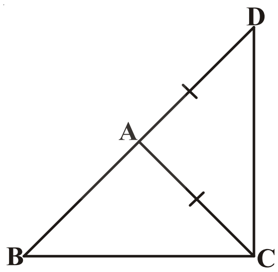
Now draw a triangle ABC and measure its sides. Find the sum of the sides AB + BC, BC + AC and AC + AB, compare it with the length of the third side. What do you observe?

You will observe that AB + BC > AC,

BC + AC > AB and AC + AB > BC.

Repeat this activity with other triangles and with this you can arrive at the following theorem:

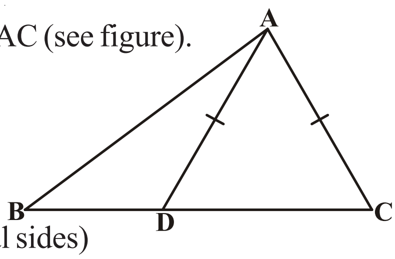
Theorem-7.8 : The sum of any two sides of a triangle is greater than the third side.



In adjacent figure, observe that the side BA of ΔABC has been produced to a point D such that AD = AC. Can you show that $\angle BCD > \angle BDC$ and BA + AC > BC? Have you arrived at the proof of the above theorem.

Let us take some examples based on these results.

Example-15. D is a point on side BC ΔABC such that AD = AC (see figure). Show that AB > AD.



Solution : In ΔDAC,

AD = AC (Given)

So, $\angle ADC = \angle ACD$ (Angles opposite to equal sides)

Now, $\angle ADC$ is an exterior angle for ΔABD.

So, $\angle ADC > \angle ABD$

or, $\angle ACD > \angle ABD$

or, $\angle ACB > \angle ABC$

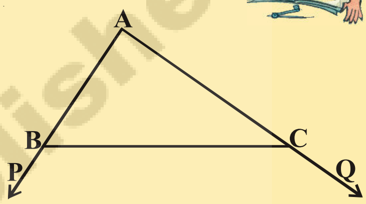
So, $AB > AC$ (Side opposite to larger angle in ΔABC)

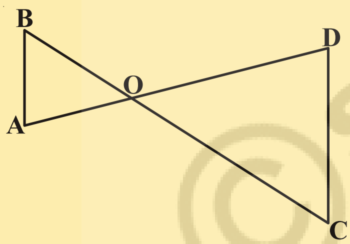
or, $AB > AD$ $(AD = AC)$

EXERCISE - 7.4

1. Show that in a right angled triangle, the hypotenuse is the longest side.

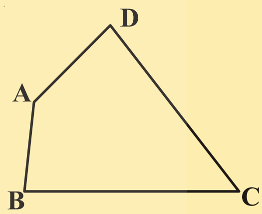
2. In adjacent figure, sides AB and AC of ΔABC are extended to points P and Q respectively. Also, $\angle PBC < \angle QCB$. Show that $AC > AB$.

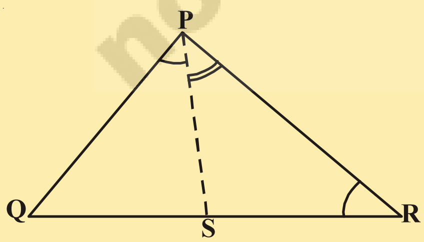




3. In adjacent figure, $\angle B < \angle A$ and $\angle C < \angle D$. Show that $AD < BC$.

4. AB and CD are respectively the smallest and longest sides of a quadrilateral ABCD (see adjacent figure). Show that $\angle A > \angle C$ and $\angle B > \angle D$.





5. In adjacent figure, $PR > PQ$ and PS bisects $\angle QPR$. Prove that $\angle PSR > \angle PSQ$.

6. If two sides of a triangle measure 4cm and 6cm find all possible measurements (positive Integers) of the third side. How many distinct triangles can be obtained?

7. Try to construct a triangle with 5cm, 8cm and 1cm. Is it possible or not? Why? Give your justification?

WHAT WE HAVE DISCUSSED

- Figures which are identical i.e. having same shape and size are called congruent figures.
- Three independent elements to make a unique triangle.
- Two triangles are congruent if the sides of one triangle are equal to the sides of another triangle and the corresponding angles in the two triangles are equal.
- Also, there is a one-one correspondence between the vertices.
- In Congruent triangles corresponding parts are equal and we write in short 'CPCT' for corresponding parts of congruent triangles.
- SAS congruence rule: Two triangles are congruent if two sides and the included angle of one triangle are equal to the corresponding two sides and the included angle of the other triangle.
- ASA congruence rule: Two triangles are congruent if two angles and the included side of one triangle are equal to two angles and the included side of other triangle.
- Angles opposite to equal sides of an isosceles triangle are equal.
- Conversely, sides opposite to equal angles of a triangle are equal.
- SSS congruence rule: If three sides of one triangle are equal to the three sides of another triangle, then the two triangles are congruent.
- RHS congruence rule: If in two right triangles the hypotenuse and one side of one triangle are equal to the hypotenuse and corresponding side of the other triangle, then the two triangles are congruent.
- If two sides of a triangle are unequal, the angle opposite to the longer side is larger.
- In any triangle, the side opposite to the larger angle is longer.
- The sum of any two sides of a triangle is greater than the third side.