15

Proofs in Mathematics

15.1 Introduction

We come across many statements in our daily life. We gauge the worth of each statement. Some statements we consider to be appropriate and true and some we dismiss. There are some we are not sure of. How do we make these judgements? In case there is a statement of conflict about loans or debts. You want to claim that bank owes your money then you need to present documents as evidence of the monetary transaction. Without that people would not believe you. If we think carefully we can see that in our daily life we need to prove if a statement is true or false. In our conversations in daily life we sometimes do not consider it relevant to prove or check statements and accept them without serious examination. That however is not at all accepted in mathematics. Consider the following:

1. The sun rises in the east.
2. $3 + 2 = 5$
3. New York is the capital of USA.
4. $4 > 8$
5. How many siblings do you have?
6. Goa has better football team than Bengal.
7. Rectangle has 4 lines of symmetry.
8. $x + 2 = 7$
9. Please come in.
10. The probability of getting two consecutive 6's on throws of a 6 sided dice is ?
11. How are you?
12. The sun is not stationary but moving at high speed all the time.
13. $x < y$
14. Where do you live?

Out of these some sentences we know are false. For example, 4>8. Similarly we know that at present New York is not the capital of USA. Some we can say are correct from our present knowledge. These include "sun rises in the east." The probability............"

The Sun is not stationary......................

Besides those there are some other sentences that are true for some known cases but not true for other cases, for example $x + 2 = 7$ is true only when $x = 5$ and $x < y$ is only true for those values of x and y where x is less than y.

Look at the other sentences which of them are clearly false or clearly true. These are statements. We say these sentences that can be judged on some criteria, no matter by what process for their being true or false are statements.

Think about these:

1. Please ignore this notice.....
2. The statement I am making is false.
3. This sentence has some words.
4. You may find water on the moon.

Can you say whether these sentences are true or false? Is there any way to check them being true or false?

Look at the first sentence, if you ignore the notice, you do that because it tells you to do so. If you do not ignore the notice, then you have paid some attention to it. So you can never follow it and being an instruction it cannot be judged on a true/false scale. 2^{nd} and 3^{rd} sentences are talking about themselves. 4th sentence have words that show only likely or possibility and hence ambiguity of being on both sides.

The sentences which are talking about themselves and the sentences with possibility are not statements.

DO THIS

Make 5 more sentences and check whether they are statements or not. Give reasons.

15.2 MATHEMATICAL STATEMENTS

We can write infinetely large number of sentences. You can think of the kind of sentences you use and whether you can count the number of sentences you can speak? Not all these however can be judged on the criteria of false and true. For example, consider, please come in. Where do you live? Such sentences can also be very large in number.

All these the sentences are not statements. Only those that can be judged to be true or false but not both are statements. The same is true for mathematical statements. A mathemtical statement can not be ambiguous. In mathematics a statement is only acceptable if it is either true or false. Consider the following sentences:

1. 3 is a prime number.
2. Product of two odd integers is even.
3. For any real number x; $4x + x = 5x$
4. The earth has one moon.
5. Ramu is a good driver.
6. Bhaskara has written a book "Leelavathi".
7. All even numbers are composite.
8. A rhombus is a square.
9. $x > 7$.
10. 4 and 5 are relative primes.

11. Silver fish is made of silver.
12. Humans are meant to rule the earth.
13. Fo any real number x, $2x > x$.
14. Havana is the capital of Cuba.

Which of these are mathematical and which are not mathematical statements?

15.3 VERIFYING THE STATEMENTS

Let us consider some of the above sentences and discuss them as follows:

Example-1. We can show that (1) is true from the definition of a prime number.

Which of the sentences from the above list are of this kind of statements that we can prove mathematically? (Try to prove).

Example-2. "Product of two odd integers is even". Consider 3 and 5 as the odd integers. Their product is 15, which is not even.

Thus it is a statement which is false. So with one example we have showed this. Here we are able to verify the statement using an example that runs counter to the statement. Such an example, that counters a statement is called a counter example.

TRY THIS

Which of the above statements can be tested by giving a counter example ?

Example-3. Among the sentences there are some like **"Humans are meant to rule the earth"** or **"Ramu is a good driver**."

These sentences are ambiguous sentences as the meaning of ruling the earth is not specific. Similarly, the definition of a good driver is not specified.

We therefore recognize that a 'mathematical statement' must comprise of terms that are understood in the same way by everyone.

Example-4. Consider some of the other sentences like

The earth has one Moon.

Bhaskara has written the book "Leelavathi"

Think about how would you verify these to consider as statements?

These are not ambiguous statements but needs to be tested. They require some observations or evidences. Besides, checking this statement cannot be based on using previously known results. The first sentence require observations of the solar system and more closely of the earth. The second sentence require other documents, references or some other records.

Mathematical statements are of a distinct nature from these. They cannot be proved or justified by getting evidence while as we have seen, they can be disproved by finding an example counter to the statement. In the statement for any real number $2x > x$, we can take $x = -1$ or $-\frac{1}{2}$ .... and disprove the statement by giving counter example. You might have also noticed that $2x > x$ is true with a condition on x i.e. x belong to set N.

Example-5. Restate the following statements with appropriate conditioins, so that they become true statements.

i. For every real number x, $3x > x$.

ii. For every real number x, $x^2 \geq x$.

iii. If you divide a number by two, you will always get half of that number.

iv. The angle subtended by a chord of a circle at a point on the circle is 90°.

v. If a quadrilateral has all its sides equal, then it is a square.

Solution :

i. If $x > 0$, then $3x > x$.

ii. If $x \leq 0$ or $x \geq 1$, then $x^2 \geq x$.

iii. If you divide a number other than 0 by 2, then you will always get half of that number.

iv. The angle subtended by a diameter of a circle at a point on the circle is 90°.

v. If a quadrilateral has all its sides and interior angles equal, then it is a square.

EXERCISE - 15.1

1. State whether the following sentences are always true, always false or ambiguous. Justify your answer.

   i. There are 27 days in a month.

   ii. Makarasankranthi falls on a Friday.

   iii. The temperature in Hyderabad is 2°C.

   iv. The earth is the only planet where life exist.

   v. Dogs can fly.

   vi. February has only 28 days.

2. State whether the following statements are true or false. Give reasons for your answers.

   i. The sum of the interior angles of a quadrilateral is 350°.

   ii. For any real number x, $x^2 \geq 0$.

   iii. A rhombus is a parallelogram.

   iv. The sum of two even numbers is even.

   v. Square numbers can be written as the sum of two odd numbers.

3. Restate the following statements with appropriate conditions, so that they become true statements.

   i. All numbers can be represented in prime factorization.

   ii. Two times a real number is always even.

   iii. For any x, $3x + 1 > 4$.

   iv. For any x, $x^3 \geq 0$.

   v. In every triangle, a median is also an angle bisector.

4. Disprove, by finding a suitable counter example, the statement $x^2 > y^2$ for all $x > y$.

15.4 REASONING IN MATHEMATICS

We human beings are naturally curious. This curiosity makes us to interact with the world. What happens if we push this? What happens if we stuck our finger in that? What happens if we make various gestures and expressions? From this experimentation, we human beings begin to form a more or less consistent picture of the way that the physical world behaves. Gradually, in all manner of situations, we make a shift from

'What happens if.....?' to *'this will happen if'*

The experimentation moves on to the exploration of new ideas and the refinement of our world view of previously understood situations. This description of the playtime pattern very nicely models the concept of 'making and testing hypothesis.' It follows this pattern:

- Make some observations, Collect some data based on the observations.
- Draw conclusion (called a 'hypothesis') which will explain the pattern of the observations.
- Test out hypothesis by making some more targeted observations.

So, we have

- A **hypothesis** is a statement or idea which gives an explanation to a series of observations.

Sometimes, following observation, a hypothesis will clearly need to be refined or rejected. This happens if a *single* contradictory observation occurs. In general we use word conjecture in mathematics instead of hypothesis. You will learn the similarities and difference between these two in your higher classes.

15.4.1 Using deductive reasoning in hypothesis testing

There is often confusion between the ideas surrounding proof, which is mathematics, and making and testing an experimental hypothesis, which is science. The difference is rather simple:

- Mathematics is based on *deductive reasoning* : a proof is a logical deduction from a set of clear inputs.
- Science is based on *inductive reasoning* : hypotheses are strengthened or rejected based on an accumulation of experimental evidence.

Of course, to be good at science, you need to be good at deductive reasoning, although experts at deductive reasoning need not be mathematicians.

Detectives, such as Sherlock Holmes and Hercule Poirot, are such experts : they collect evidence from a crime scene and then draw logical conclusions from the evidence to support the hypothesis that, for example, person M. committed the crime. They use this evidence to create sufficiently compelling deductions to support their hypothesis *beyond reasonable doubt*. The key word here is 'reasonable'.

15.4.2 Deductive Reasoning

The main logical tool used in establishing the truth of an **unambiguous** statement is *deductive reasoning*. To understand what deductive reasoning is all about, let us begin with a puzzle for you to solve.

You are given four cards. Each card has a number printed on one side and a letter on the other side.

A	V	8	5

Suppose you are told that these cards follow the rule:

"If a card has an odd number on one side, then it has a vowel on the other side."

What is the **smallest number** of cards you need to turn over to check if the rule is true?

Of course, you have the option of turning over all the cards and checking. But can you manage with turning over a fewer number of cards?

Notice that the statement mentions that a card with an odd number on one side has a vowel on the other. It does not state that a card with a vowel on one side must have an odd number on the other side. That may or may not be so. The rule also does not state that a card with an even number on one side must have a consonant on the other side. It may or may not.

So, do we need to turn over $\boxed{A}$? No! Whether there is an even number or an odd number on the other side, the rule still holds.

What about $\boxed{8}$? Again we do not need to turn it over, because whether there is a vowel or a consonant on the other side, the rule still holds.

But you do need to turn over $\boxed{V}$ and $\boxed{5}$. if V has an odd number on the other side, then the rule has been broken. Similarly, if $\boxed{5}$ has a consonant on the other side, then the rule has been broken.

The kind of reasoning we have used to solve the puzzle is called **deductive reasoning**. It is called 'deductive' because we arrive at (i.e., deduce or infer) a result or a statement from a previously established statement using logic. For example, in the puzzle above, by a series of logical arguments we deduced that we need to turn over only $\boxed{V}$ and $\boxed{5}$.

Deductive reasoning also helps us to conclude that a particular statement is true, because it is a special case of a more general statement that is known to be true. For example, once we prove that the product of two even numbers is always even, we can immediately conclude (without computation) that 56702×19992 is even simply because 56702 and 19992 are even.

Consider some other examples of deductive reasoning:

i. If a number ends in '0' it is divisible by 5. 30 ends in 0.

From the above two statements we can deduce that 30 is divisible by 5 because it is given that the number ends in 0 is divisible by 5.

ii. Some singers are poets. All lyrists are Poets.

Here the deduction based on two statement is wrong. (Why?) All lyricist are poets (wrong). Because we are not sure about it. There are three posibilities (i) all lyricists could be poets, (ii) few could be poets or (iii) none of the lyricists is a poet.

You may come to a conclusion that if - then conditional statement comes into deductive reasoning. In mathematics we use this reasoning a lot like if linear pair of angles are 180°. Then only the sum of angles in a triangle is equal to 180°. Like wise if we are using decimal number system to write a number 5. If we use the binary system we represent the quantity by 101.

Unfortunately we do not always use correct reasoning in our daily life. We often come to many conclusions based on faulty reasoning. For example, if your friend does not talk to you one day, then you may conclude that she is angry with you. While it may be true that "if she is angry with you she will not talk to me", it may also be true that "if she is busy and she will not talk to me. Why don't you examine some conclusions that you have arrived at in your day-to-day existence, and see if they are based on valid or faulty reasoning?

EXERCISE - 15.2

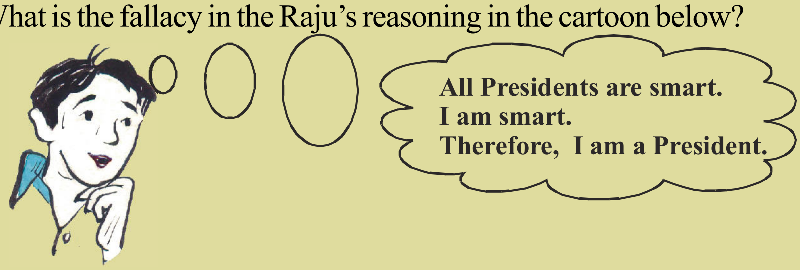
1. Use deductive reasoning to answer the following:

   i. Human beings are mortal. Jeevan is a human being. Based on these two statements, what can you conclude about Jeevan ?

   ii. All Telugu people are Indians. X is an Indian. Can you conclude that X belongs to Telugu people.

   iii. Martians have red tongues. Gulag is a Martian. Based on these two statements, what can you conclude about Gulag?

   iv. What is the fallacy in the Raju's reasoning in the cartoon below?



2. Once again you are given four cards. Each card has a number printed on one side and a letter on the other side. Which are the only two cards you need to turn over to check whether the following rule holds?

   "If a card has a consonant on one side, then it has an odd number on the other side."

B	3	U	8

3. Think of this puzzle What do you need to find a chosen number from this square?

   Four of the clues below are true but do nothing to help in finding the number.

   Four of the clues are necessary for finding it.

   Here are eight clues to use:

   a. The number is greater than 9.
   b. The number is not a multiple of 10.
   c. The number is a multiple of 7.
   d. The number is odd.
   e. The number is not a multiple of 11.
   f. The number is less than 200.
   g. Its ones digit is larger than its tens digit.
   h. Its tens digit is odd.

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

   What is the number?

   Can you sort out the four clues that help and the four clues that do not help in finding it?

   First follow the clues and strike off the number which comes out from it.

   Like - from the first clue we come to know that the number is not from 1 to 9. (strike off numbers from 1 to 9).

   After completing the puzzle, see which clue is important and which is not?

15.5 THEOREMS, CONJECTURES AND AXIOMS

So far we have discussed statements and how to check their validity. In this section, you will study how to distinguish between the three different kinds of statements Mathematics is built up from, namely, a theorem, a conjecture and an axiom.

You have already come across many theorems before. So, what is a theorem? A mathematical statement whose truth has been established (proved) is called a *theorem*. For example, the following statements are theorems.

Theorem-15.1 : The sum of the interior angles of a triangle is 180°.

Theorem-15.2 : The product of two odd natural numbers is odd.

Theorem-15.3 : The product of any two consecutive even natural numbers is divisible by 4.

A *conjecture* is a statement which we believe is true, based on our mathematical understanding and experience, that is, our mathematical intuition. The conjecture may turn out to be true or false. If we can prove it, then it becomes a theorem. Mathematicians often come up with conjectures by looking for patterns and making intelligent mathematical guesses. Let us look at some patterns and see what kind of intelligent guesses we can make.

Raju noticed, while studying some cube numbers, that "if you take three consecutive whole numbers and multiply them together and then add the middle number of the three, you get the middle number cubed"; *e.g.*, 3, 4, 5, gives $3 \times 4 \times 5 + 4 = 64$, which is a perfect cube. Does this always work? Take some more consecutive numbers and check it.

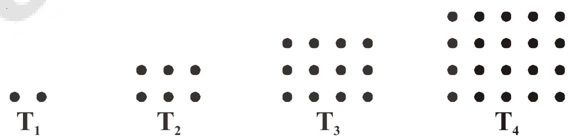
Rafi took 6, 7, 8 and checked this conjecture. Here 7 is the middle term so according to the rule $6 \times 7 \times 8 + 7 = 343$, which is also a perfect cube. Try to generalize it by taking numbers as n, $n+1$, $n+2$. See other example:

Example-6. The following geometric arrays suggest a sequence of numbers.

(a) Find the next three terms.

(b) Find the 100^{th} term.

(c) Find the n^{th} term.



The dots here are arranged in such a way that they form a rectangle. Here $T_1 = 2$, $T_2 = 6$, $T_3 = 12$, $T_4 = 20$ and so on. Can you guess what T_5 is? What about T_6? What about T_n?

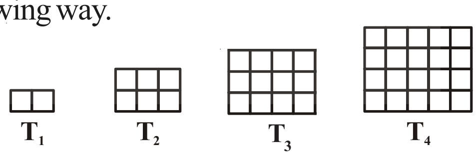
Make a conjecture about T_n.

It might help if you redraw them in the following way.

Solution :

T_1	T_2	T_3	T_4	T_5	T_6
2	6	12	20	?	.....

+4 +6 +8 +10



So, $T_5 = T_4 + 10 = 20 + 10 = 30 = 5 \times 6$

$T_6 = T_5 + 12 = 30 + 12 = 42 = 6 \times 7$ ..... Try for T_7?

$T_{100} = 100 \times 101 = 10,100$

$T_n = n \times (n+1) = n^2 + n$

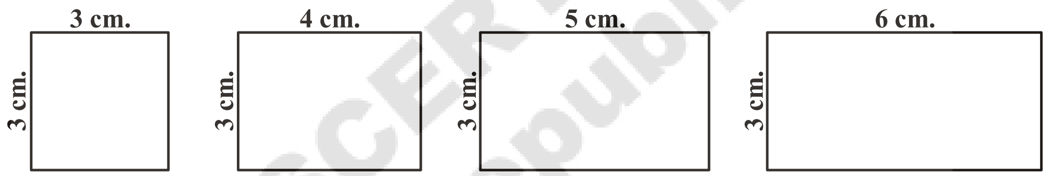
This type of reasoning which is based on examining a variety of cases or sets of data, discovering patterns and forming conclusions is called ***inductive reasoning***. Inductive reasoning is very helpful technique for making conjecture.

Gold bach the renounced mathematician, observed a pattern:

$6 = 3 + 3$	$8 = 3 + 5$	$10 = 3 + 7$
$12 = 5 + 7$	$14 = 11 + 3$	$16 = 13 + 3 = 11 + 5$

From the pattern Gold bach in 1743 reasoned that every even number greater than 4 can be written as the sum of two primes (not necessarily distinct primes). His conjecture has not been proved to be true or false so far. Perhaps you will prove that this result is true or false and will become famous.

But just by looking few patterns some time lead us to a wrong conjecture like: in class 8th Janvi and Kartik while studing Area and Perimeter chapter..... observed a pattern

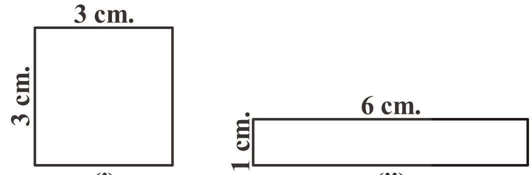


	(i)	(ii)	(iii)	(iv)
Perimeter :	12 cm.	14 cm.	16 cm.	18 cm.
Area :	9 cm^2	12 cm^2	15 cm^2	18 cm^2

and stated a conjecture that when the perimeter of the rectangle increases the area will also increase. What do you think? Are they right?

While working on this pattern.

Inder drew some rectangles and disproved the conjecture stated by Janvi and Kartik.

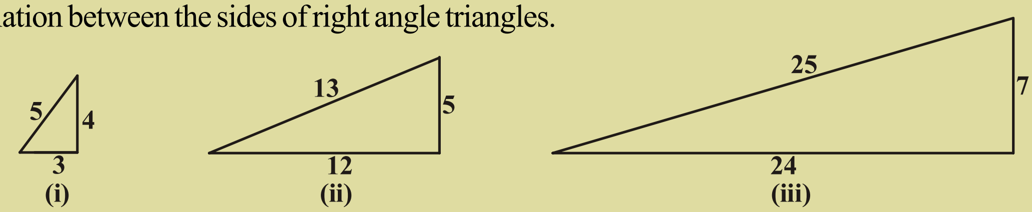


	(i)	(ii)
Perimeter :	12 cm.	14 cm.
Area :	9 cm^2	6 cm^2

Do you understand that while making a conjecture we have to look all the possibilities.

Try This

Envied by the popularity of Pythagoras his younger brother claimed a different relation between the sides of right angle triangles.



Liethagoras Theorem : In any right angle triangle the square of the smallest side equals the sum of the other sides.

Check this conjucture, whether it is right or wrong.

You might have wondered - do we need to prove every thing we encounter in mathematics and if not, why not?

In mathematics some statements are assumed to be true and are not proved, these are self-evident truths' which we take to be true without proof. These statements are called *axioms*. In chapter 3, you would have studied the axioms and postulates of Euclid. (We do not distinguish between axioms and postulates these days generally we use word postulate in geometry).

For example, the first postulate of Euclid states:

A straight line may be drawn from any point to any other point.

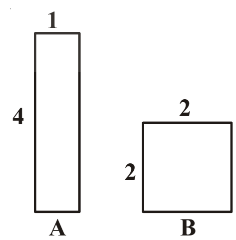
And the third postulate states:

A circle may be drawn with any centre and any radius.

These statements appear to be perfectly true and Euclid assumed them to be true. Why? This is because we cannot prove everything and we need to start somewhere, we need some statements which we accept as true and then we can build up our knowledge using the rules of logic based on these axioms.

You might then wonder why don't we just accept all statements to be true when they appear self evident. There are many reasons for this. Very often our intuition can be wrong, pictures or patterns can deceive and the only way to be sure that something is true is to prove it. For example, many of us believe that if a number is added to another number, the result will be large than the numbers. But we know that this is not always true : for example 5 + (-5) = 0, which is smaller than 5.

Also, look at the figures. Which has bigger area ?



It turns out that both are of exactly the same area, even though B appears bigger.

You might then wonder, about the validity of axioms. Axioms have been chosen based on our intuition and what appears to be self-evident. Therefore, we expect them to be true. However, it is possible that later on we discover that a particular axiom is not true. What is a safeguard against this possibility? We take the following steps:

i. Keep the axioms to the bare minimum. For instance, based only on axioms and five postulates of Euclid, we can derive hundreds of theorems.

ii. Make sure the axioms are consistent.

We say a collection of axioms is *inconsistent*, if we can use one axiom to show that another axiom is not true. For example, consider the following two statements. We will show that they are inconsistent.

Statement-1 : No whole number is equal to its successor.

Statement-2 : A whole number divided by zero is a whole number.

(Remember, **division by zero is not defined**. But just for the moment, we assume that it is possible, and see what happens.)

From Statement-2, we get $\frac{1}{0} = a$, where a is some whole number. This implies that, 1=0. But this disproves Statement-1, which states that no whole number is equal to its successor.

iii. A false axiom will, sooner or later, result into contradiction. We say that *there is a contradiction, when we find a statement such that, both the statement and its negation are true.* For example, consider Statement-1 and Statement-2 above once again.

From Statement-1, we can derive the result that $2 \neq 1$.

Let $x = y$

$x \times x = xy$

$x^2 = xy$

$x^2 - y^2 = xy - y^2$

$(x+y)(x-y) = y(x-y)$ From Statement-2, we can cancel $(x - y)$ from both the sides.

$x + y = y$

But $x = y$

so $x + x = x$

or $2x = x$

$2 = 1$

So we have both the statements $2 \neq 1$ and its negation, $2 = 1$ are true. This is a contradiction. The contradiction arose because of the false axiom, that a whole number divided by zero is a whole number.

So, the statement we choose as axioms require a lot of thought and insight. We must make sure they do not lead to inconsistencies or logical contradictions. Moreover, the choice of axioms themselves, sometimes leads us to new discoveries.

We end the section by recalling the differences between an axiom, a theorem and a conjecture. An **axiom** is a mathematical statement which is assumed to be true without proof; a **conjecture** is a mathematical statement whose truth or falsity is yet to be established; and a **theorem** is a mathematical statement whose truth has been logically established.

EXERCISE - 15.3

1. (i) Take any three consecutive odd numbers and find their product;

   for example, $1 \times 3 \times 5 = 15$, $3 \times 5 \times 7 = 105$, $5 \times 7 \times 9$ - .....

   (ii) Take any three consecutive even numbers and add them, say,

   $2 + 4 + 6 = 12$, $4 + 6 + 8 = 18$, $6 + 8 + 10 = 24$, $8 + 10 + 12 = 30$ and so on.

   Is there any pattern you can guess in these sums? What can you conjecture about them?

2. Go back to Pascal's triangle.

   Line-1 : $1 = 11^0$

   Line-2 : $11 = 11^1$

   Line-3 : $121 = 11^2$

   Make a conjecture about Line-4 and Line-5.

   Does your conjecture hold? Does your conjecture hold for Line-6 too?

   1
   1 1
   1 2 1
   1 3 3 1
   1 4 6 4 1

3. Look at the following pattern:

   i) $28 = 2^2 \times 7^1$, Total number of factors $(2+1)(1+1) = 3 \times 2 = 6$
   28 is divisible by 6 factors i.e. 1, 2, 4, 7, 14, 28

   ii) $30 = 2^1 \times 3^1 \times 5^1$, Total number of factors $(1+1)(1+1)(1+1) = 2 \times 2 \times 2 = 8$
   30 is divisible by 8 factors i.e. 1, 2, 3, 5, 6, 10, 15, 30

   Find the pattern.
   (**Hint :** Product of every prime base exponent +1)

4. Look at the following pattern:

   $1^2 = 1$

   $11^2 = 121$

   $111^2 = 12321$

   $1111^2 = 1234321$

   $11111^2 = 123454321$

Make a conjecture about each of the following:

$111111^2 =$

$1111111^2 =$

Check if your conjecture is true.

5. List five axioms (postulates) used in this book.

6. In a polynomial $p(x) = x^2 + x + 41$ put different values of x and find $p(x)$. Can you conclude after putting different values of x that $p(x)$ is prime for all. Is x an element of N? Put $x = 41$ in $p(x)$. Now what do you find?

15.6 What is a Mathematical Proof?

Before you study proofs in mathematics, you are mainly asked to verify statements.

For example, you might have been asked to verify with examples that "the product of two odd numbers is odd". So you might have picked up two random odd numbers, say 15 and 2005 and checked that $15 \times 2005 = 30075$ is odd. You might have done so for many more examples.

Also, you might have been asked as an activity to draw several triangles in the class and compute the sum of their interior angles. Apart from errors due to measurement, you would have found that the interior angles of a triangle add up to 180°.

What is the flaw in this method? There are several problems with the process of verification. While it may help you to make a statement you believe is true, you cannot be *sure* that it is true in *all* cases. For example, the multiplication of several pairs of even numbers may lead us to guess that the product of two even numbers is even. However, it does not ensure that the product of all pairs of even numbers is even. You cannot physically check the products of all possible pairs of even numbers because they are endless. Similarly, there may be some triangles which you have not yet drawn whose interior angles do not add up to 180°.

Moreover, verification can often be misleading. For example, we might be tempted to conclude from Pascal's triangle (Q.2 of Exercise), based on earlier verification, that $11^5 = 15101051$. But in fact $11^5 = 161051$.

So, you need another approach that does not depend upon verification for some cases only. There is another approach, namely 'proving a statement'. A process which can establish the truth of a mathematical statement based purely on logical arguments is called a *mathematical proof*.

To make a mathematical statement false, we just have to produce a single counter-example. So while it is not enough to establish the validity of a mathematical statement by checking or verifying it for thousands of cases, it is enough to produce one counter example to *disprove* a statement.

Let us look what should be our procedure to prove.

i. First we must understand clearly, what is required to prove, then we should have a rough idea how to proceed.

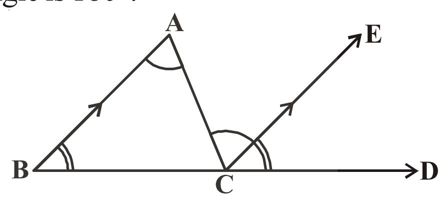
ii. A proof is made up of a successive sequence of mathematical statements. Each statement is a proof logically deduced from a previous statement in the proof or from a theorem proved earlier or an axiom or our hypothesis and what is given.

iii. The conclusion of a sequence of mathematically true statements laid out in a logically correct order should be what we wanted to prove, that is, what the theorem claims.

To understand that, we will analyse the theorem and its proof. You have already studied this theorem in chapter-4. We often resort to diagrams to help us to prove theorems, and this is very important. However, each statement in proof has to be established using only logic. Very often we hear or said statement like those two angles must be 90°, because the two lines look as if they are perpendicular to each other. Beware of being deceived by this type of reasoning.

Theorem-15.4 : The sum of three interior angles of a triangle is 180°.

Proof : Consider a triangle ABC.



We have to prove that

$\angle ABC + \angle BCA + \angle CAB = 180°$

Construct a line CE parallel to BA through C and produce line BC to D.

CE is parallel to BA and AC is transversal.

So, $\angle CAB = \angle ACE$, which are alternate angles. ..... (1)

Similarly, $\angle ABC = \angle DCE$ which are corresponding angles. ..... (2)

adding eq. (1) and (2) we get

$\angle CAB + \angle ABC = \angle ACE + \angle DCE$ ..... (3)

add $\angle BCA$ on both the sides.

We get, $\angle ABC + \angle BCA + \angle CAB = \angle DCE + \angle BCA + \angle ACE$ ..... (4)

But $\angle DCE + \angle BCA + \angle ACE = 180°$, since they form a straight angle. ..... (5)

Hence, $\angle ABC + \angle BCA + \angle CAB = 180°$

Now, we see how each step has been logically connected in the proof.

Step-1: Our theorem is concerned with a property of triangles. So we begin with a triangle ABC.

Step-2: The construction of a line CE parallel to BA and producing BC to D is a vital step to proceed so that to be able to prove the theorem.

Step-3: Here we conclude that $\angle CAB = \angle ACE$ and $\angle ABC = \angle DCE$, by using the fact that CE is parallel to BA (construction), and previously known theorems, which states that if two parallel lines are intersected by a transversal, then the alternate angles and corresponding angles are equal.

Step-4: Here we use Euclid's axiom which states that "if equals are added to equals, the wholes are equal" to deduce $\angle ABC + \angle BCA + \angle CAB = \angle DCE + \angle BCA + \angle ACE$.

That is, the sum of three interior angles of a triangle is equal to the sum of angles on a straight line.

Step-5: Here in concluding the statement we use Euclid's axiom which states that "things which are equal to the same thing are equal to each other" to conclude that

$$\angle ABC + \angle BCA + \angle CAB = \angle DCE + \angle BCA + \angle ACE = 180°$$

This is the claim made in the theorem we set to prove.

You now prove theorem-15.2 and 15.3 without analysing them.

Theorem-15.5 : The product of two odd natural numbers is odd.

Proof: Let x and y be any two odd natural numbers.

We want to prove that xy is odd.

Since x and y are odd, they can be expressed in the form $x = (2m - 1)$, for some natural number m and $y = 2n - 1$, for some natural number n.

Then, $xy = (2m - 1)(2n - 1)$

$= 4mn - 2m - 2n + 1$

$= 4mn - 2m - 2n + 2 - 1$

$= 2(2mn - m - n + 1) - 1$

Let $2mn - m - n + 1 = l$, any natural number, replace it in the above equation.

$= 2l - 1, l \in N$

This is definitely an odd number.

Theorem-15.6 : The product of any two consecutive even natural numbers is divisible by 4.

Any two consecutive even number will be of the form $2m$, $2m + 2$, for some natural number n. We have to prove that their product $2m\,(2m + 2)$ is divisible by 4. (Now try to prove this yourself).

We conclude this chapter with a few remarks on the difference between how mathematicians discover results and how formal rigorous proofs are written down. As mentioned above, each proof has a key intiative idea. Intution is central to a mathematicians' way of thinking and discovering results. A mathematician will often experiment with several routes of thought, logic and examples, before she/he can hit upon the correct solution or proof. It is only after the creative phase subsides that all the arguments are gathered together to form a proper proof.

We have discussed both inductive reasoning and deductive reasoning with some examples.

It is worth mentioning here that the great Indian mathematician Srinivasa Ramanujan used very high levels of intuition to arrive at many of his statements, whch he claimed were true. Many of these have turned out to be true and as well as known theorems.

Exercise - 15.4

1. State which of the following are mathematical statements and which are not? Give reason.
   i. She has blue eyes
   ii. $x + 7 = 18$
   iii. Today is not Sunday.
   iv. For each counting number x, $x + 0 = x$
   v. What time is it?
2. Find counter examples to disprove the following statements:
   i. Every rectangle is a square.
   ii. For any integers x and y, $\sqrt{x^2 + y^2} = x + y$
   iii. If n is a whole number then $2n^2 + 11$ is a prime.
   iv. Two triangles are congruent if all their corresponding angles are equal.
   v. A quadrilateral with all sides are equal is a square.
3. Prove that the sum of two odd numbers is even.
4. Prove that the product of two even numbers is an even number.

5. Prove that if x is odd, then x^2 is also odd.

6. Examine why they work ?

i. Choose a number. Double it. Add nine. Add your original number. Divide by three. Add four. Subtract your original number. Your result is seven.

ii. Write down any three-digit number (for example, 425). Make a six-digit number by repeating these digits in the same order (425425). Your new number is divisible by 7, 11, and 13.

WHAT WE HAVE DISCUSSED

1. The sentences that can be judged on some criteria, no matter by what process for their being true or false are statements.
2. Mathematical statements are of a distinct nature from general statements. They can not be proved or justified by getting evidence while they can be disproved by finding a counter example.
3. Making mathematical statements through observing patterns and thinking of the rules that may define such patterns.

   A hypothesis is a statement of idea which gives an explanation to a sense of observation.
4. A process which can establish the truth of a mathematical statement based purely on logical arguments is called a mathematical proof.
5. Axioms are statements which are assumed to be true without proof.
6. A conjecture is a statement we believe is true based on our mathematical intution, but which we are yet to prove.
7. A mathematical statement whose truth has been established or proved is called a theorem.
8. The prime logical method in proving a mathematical statement is deductive reasoning.
9. A proof is made up of a successive sequence of mathematical statements.
10. Begining with given (Hypothesis) of the theorem and arrive at the conclusion by means of a chain of logical steps is mostly followed to prove theorems.
11. The proof in which, we start with the assumption contrary to the conclusion and arriving at a contradiction to the hypothesis is another way that we establish the original conclusion is true is another type of deductive reasoning.
12. The logical tool used in establishment the truth of an unambiguious statements to deductive reasoning.
13. The resoning which is based on examining of variety of cases or sets of data discovering pattern and forming conclusion is called Inductive reasoning.