13 Geometrical Constructions

13.1 Introduction

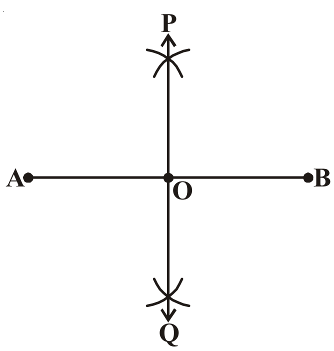
To construct geometrical figures, such as a line segment, an angle, a triangle, a quadrilateral etc., some basic geometrical instruments are needed. You must be having a geometry box which contains a graduated ruler (Scale) a pair of set squares, a divider, a compass and a protractor.

Generally, all these instruments are needed in drawing. A geometrical construction is the process of drawing a geometrical figure using only two instruments - an ungraduated ruler and a compass. We have mostly used ruler and compass in the construction of triangles and quadrilaterals in the earlier classes. In construction where some other instruments are also required, you may use a graduated scale and protractor as well. There are some constructions that cannot be done straight away. For example, when there are 3 measures available for the triangle, they may not be used directly. We will see in this chapter, how to extract the needed values and complete the required shape.

13.2 Basic Constructions

You have learnt how to construct (i) the perpendicular bisector of a line segment, (ii) angle bisector of 30°, 45°, 60°, 90° and 120° or of a given angle, in the lower classes. However the reason for these constructions were not discussed. The objective of this chapter is to give the process of necessary logical proofs to all those constructions.

13.2.1 To Construct the perpendicular bisector of a given line segment.



Example-1. Draw the perpendicular bisector of a given line segment AB and write justification.

Solution : Steps of construction.

Steps 1 : Draw the line segment AB

Step 2 : Taking A centre and with radius more than $\frac{1}{2}\overline{AB}$, draw an arc on either side of the line segment AB.

Step 3 : Taking 'B' as centre, with the same radius as above, draw arcs so that they intersect the previously drawn arcs.

Step 4 : Mark these points of intersection as P and Q.

Join P and Q.

Step 5 : Let PQ intersect $\overline{AB}$ at the point O

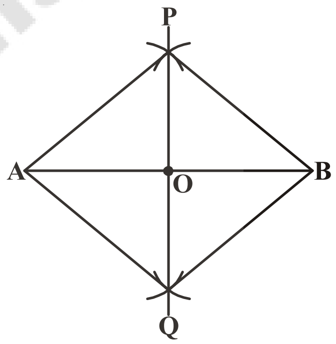
Thus the line POQ is the required perpendicular bisector of AB.

How can you prove the above construction i.e. "PQ is perpendicular bisector of AB", logically?

Draw diagram of construction and join A to P and Q; also B to P and Q.

We use the congruency of triangle properties to prove the required.

Proof :



Steps	**Reasons**
In Δ^s PAQ and ΔPBQ	Selected
AP = BP ; AQ = BQ	Equal radii
PQ = PQ	Common side
$\therefore$ $\Delta PAQ \cong \Delta PBQ$	SSS rule
So $\angle APO = \angle BPO$	CPCT (corresponding parts of congruent triangles)
Now In Δ^s APO and BPO	Selected
AP = BP	Equal radii as before
$\angle APO = \angle BPO$	Proved above
OP = OP	Common
$\therefore$ $\Delta APO \cong \Delta BPO$	SAS rule
So OA = OB and $\angle APO = \angle BPO$	CPCT
As $\angle AOP + \angle BOP = 180^\circ$	Linear pair
We get $\angle AOP = \angle BOP = \frac{180^\circ}{2} = 90^\circ$	From the above result
Thus PO, i.e. POQ is the perpendicular bisector of AB	Required to prove.

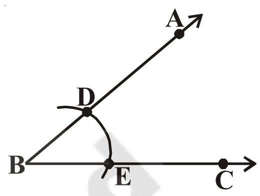
13.2.2 To construct the bisector of a given angle

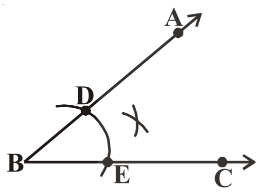
Example-2. Construct the bisector of a given angle ABC.

Solution : Steps of construction.

Step 1 : Draw the given angle ABC

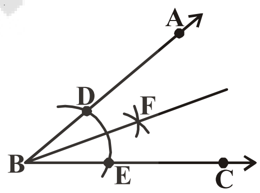
Step 2 : Taking B as centre and with any radius, draw an arc to intersect the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, at D and E respectively, as shown in the figure.





Step 3 : Taking E and D as centres draw two arcs with equal radii to intersect each other at F.

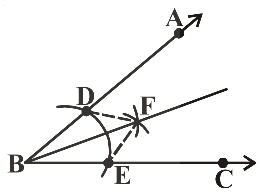
Step 4 : Draw the ray BF. It is the required bisector of $\angle ABC$.



Let us see the logical proof of above construction. Join D, F and E, F. We use congruency rule of triangles to prove the required.

Proof :

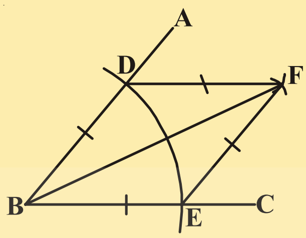
Steps	Reasons
In Δ^s BDF and ΔBEF	Selected triangles
BD = BE	radii of same arc
DF = EF	Arcs of equal radii
BF = BF	Common
$\therefore \Delta BDF \cong \Delta BEF$	SSS rule
So $\angle DBF = \angle EBF$	CPCT
Thus BF is the bisector of $\angle ABC$	Required to prove



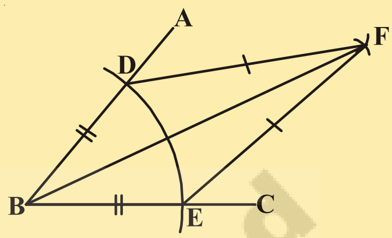
TRY THESE

Observe the sides, angles and diagonals of quadrilateral BEFD. Name the figures given below and write properties of figures.

1.



2.

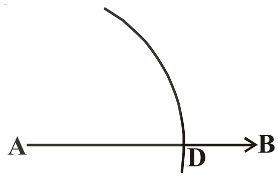


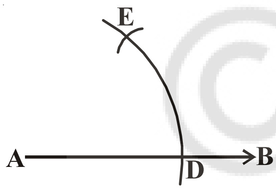
13.2.3 To construct an angle of 60° at the initial point of a given ray.

Example-3. Draw a ray AB with initial point A and construct a ray AC such that ∠BAC = 60°.

Solution : Steps of Construction

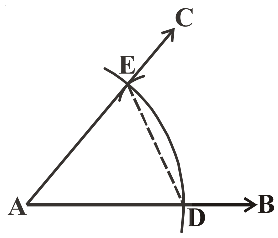
Step 1 : Draw the given ray AB and taking A as centre and some radius, draw an arc which intersects AB, say at a point D.





Step 2 : Taking D as centre and with the same radius taken before, draw an arc intersecting the previously drawn arc, say at a point E.

Step 3 : Draw a ray AC Passing through E then ∠BAC is the required angle of 60°.



Let us see how the construction is justified. Draw the figure again and join DE and prove it as follows .

Steps	Reasons
In Δ ADE	Selected
AE = AD	radii of same arc
AD = DE	Arcs of equal radius
Then AE = AD = DE	Same arc with same radii
∴ ΔADE is equilateral triangle	All sides are equal.
∠EAD = 60°	each angle of equilateral triangle.
∠BAC is same as ∠EAD	∠EAD is a part of ∠BAC.
∠BAC = 60°.	Required to prove.

TRY THIS

Draw a circle, Identify a point on it. Cut arcs on the circle with the length of the radius in succession. How many parts can the circle be divided into? Give reason.

EXERCISE - 13.1

1. Construct the following angles at the initial point of a given ray and justify the construction.

   (a) 90^{o} (b) 45^{o}

2. Construct the following angles using ruler and compass and verify by measuring them by a protractor.

   (a) 30^{o} (b) $22\frac{1}{2}^{o}$ (c) 15^{o}

   (d) 75^{o} (e) 105^{o} f) 135^{o}

3. Construct an equilateral triangle, given its side of length of 4.5 cm and justify the construction.

4. Construct an isosceles triangle, given its base and base angle and justify the construction.

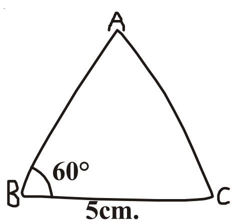
   [Hint : You can take any measure of side and angle]

13.3 CONSTRUCTION OF TRIANGLES (SPECIAL CASES)

We have so far, constructed some basic constructions and justified with proofs. Now we will construct some triangles when special type of measures are given. Recall the congruency properties of triangles such as SAS, SSS, ASA and RHS rules. You have already learnt how to construct triangles in class VII using the above rules.

You may have learnt that atleast three parts of a triangle have to be given for constructing it but not any combinations of three measures are sufficient for the purpose. For example, if two sides and an angle (not the included angle) are given, then it is not always possible to construct such a triangle uniquely. We can give several illustrations for such constructions. In such cases we have to use the given measures with desired combinations such as SAS, SSS, ASA and RHS rules.

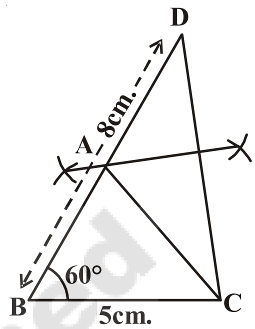
13.3.1 Construction : To construct a triangle, given its base, a base angle and sum of other two sides.



Example-4. Construct a ΔABC given BC = 5 cm., AB + AC = 8 cm. and $\angle ABC = 60^{\circ}$.

Solution : Steps of construction

Step 1 : Draw a rough sketch of ΔABC and mark the given measurements as usual.
(How can you mark $AB + AC = 8cm$?)
How can you locate third vertex A in the construction ?



Analysis : As we have $AB + AC = 8$ cm., extend BA up to D so that BD = 8 cm.

$\therefore$ BD = BA + AD = 8 cm

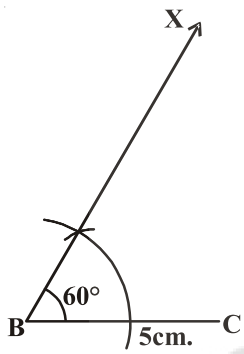
but AB + AC = 8 cm. (given)

$\because$ AD = AC

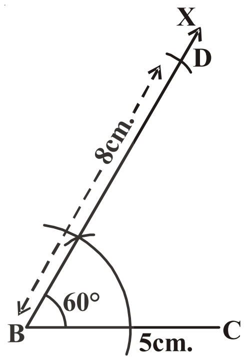
To locate A on BD what will you do ?

As A is equidistant from C and D, draw a perpendicular bisector of $\overline{CD}$ to locate A on BD.

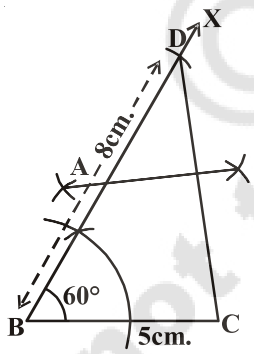
How can you prove AB + AC = BD ?



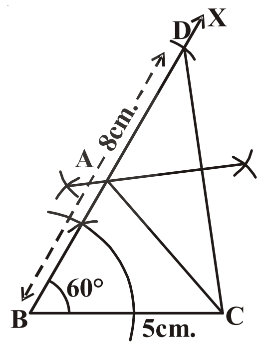
Step 2 : Draw the base $\overline{BC} = 5$ cm and construct $\angle CBX = 60^\circ$ at B



Step 3: With centre B and radius 8 cm (AB + AC = 8 cm) draw an arc on $\overrightarrow{BX}$ to intersect (meet) at D.



Step 4 : Join CD and draw a perpendicular bisector of CD to meet BD at A



Step 5 : Join AC to get the required triangle ABC.

Now, we will justify the construction.

Proof : A lies on the perpendicular bisector of $\overline{CD}$

$\because$ AC = AD

AB + AC = AB + AD

= BD

= 8 cm.

Hence ΔABC is the required triangle.

THINK, DISCUSS AND WRITE

Can you construct a triangle ABC with BC = 6 cm, $\angle B = 60^\circ$ and AB + AC = 5cm.? If not, give reasons.

13.3.2 Construction : To Construct a triangle given its base, a base angle and the difference of the other two sides.

Given the base BC of a triangle ABC, a base angle say $\angle B$ and the difference of other two sides AB – AC in case AB>AC or AC-AB, in case AB<AC, you have to construct the triangle ABC. Thus we have two cases of constructions discussed in the following examples.

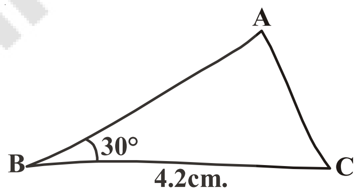
Case (i) Let AB > AC

Example-5. Construct ΔABC in which BC = 4.2 cm, $\angle B = 30^\circ$ and AB – AC = 1.6 cm

Solution : Steps of Construction

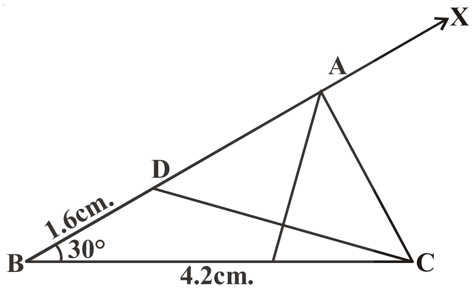
Step 1: Draw a rough sketch of ΔABC and mark the given measurements

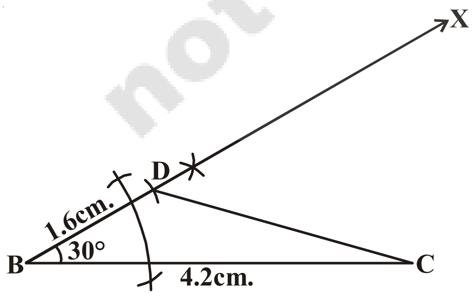
(How can you mark AB – AC = 1.6 cm ?)



Analysis : Since AB – AC = 1.6 cm and AB > AC, mark D on AB such that AD = AC Now BD = AB – AC = 1.6 cm. Join CD and draw a perpendicular bisector of CD to find the vertex A on BD produced.

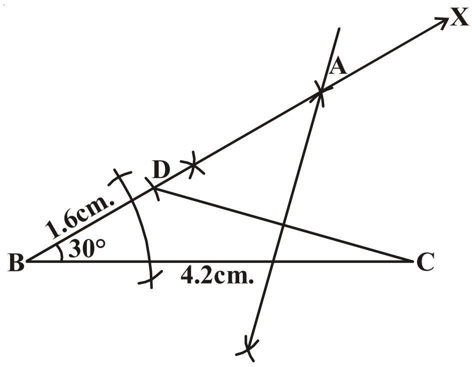
Join AC to get the required triangle ABC.



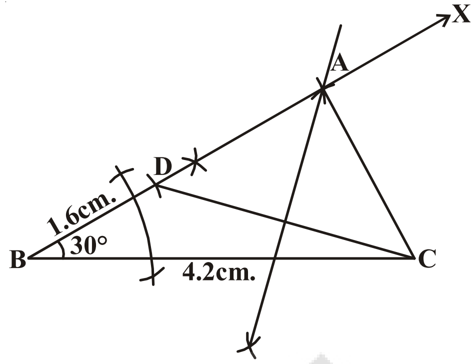


Step 2: Construct ΔBCD using S.A.S rule with measures BC = 4.2 cm $\angle B = 30^0$ and BD = 1.6 cm. (i.e. AB - AC)

Step 3 : Draw the perpendicular bisector of CD. Let it meet ray BDX at a point A.



Step 4: Join AC to get the required triangle ABC.



THINK, DISCUSS AND WRITE

Can you construct the triangle ABC with the same measures by changing the base angle $\angle C$ instead of $\angle B$? Draw a rough sketch and construct it.

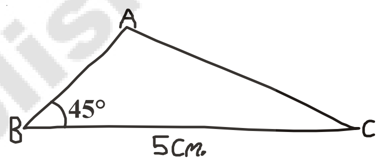
Case (ii) Let $AB < AC$

Example-6. Construct ΔABC in which $BC = 5cm$, $\angle B = 45^o$ and $AC - AB = 1.8$ cm.

Solution : Steps of Construction.

Step 1: Draw a rough sketch of ΔABC and mark the given measurements.

Analyse how $AC - AB = 1.8$ cm can be marked?



Analysis : Since $AC - AB = 1.8$ cm i.e. $AB < AC$ we have to find D on AB produced such that $AD = AC$

Now $BD = AC - AB = 1.8$ cm ($\because$ $BD = AD - AB$ and $AD = AC$)

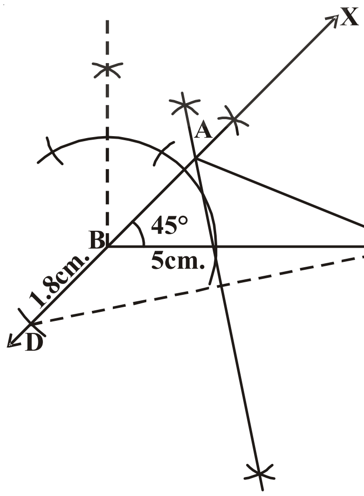
Join CD to find A on the perpendicular bisector of DC

Step 2 : Draw $BC = 5$ cm and construct $\angle CBX = 45°$

With centre B and radius 1.8 cm $(BD = AC - AB)$ draw an arc to intersect the line XB extended at a point D.

Step 3 : Join DC and draw the perpendicular bisector of DC.

Step 4 : Let it meet $\overrightarrow{BX}$ at A and join AC ΔABC is the required triangle.



Now, you can justify the construction.

Proof: In ΔABC, the point A lies on the perpendicular bisector of DC.

$\therefore$ $AD = AC$

$AB + BD = AC$

So $BD = AC - AB$

$= 1.8$ cm

Hence ΔABC is the required that triangle.

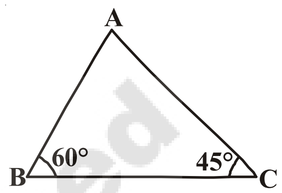
13.3.3 Construction : To construct a triangle, given its perimeter and its two base angles.

Given the base angles, say $\angle B$ and $\angle C$ and perimeter AB + BC + CA, you have to construct the triangle ABC.

Example-7. Construct a triangle ABC, in which $\angle B = 60^o$, $\angle C = 45^o$ and

AB + BC + CA = 11 cm.

Solution : Steps of construction.



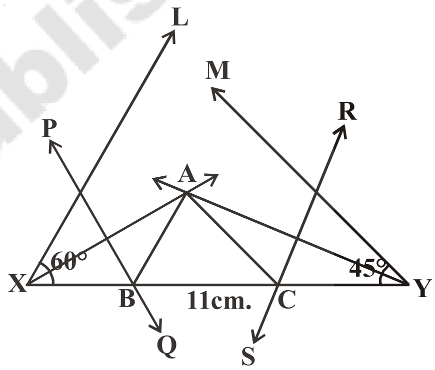
Step 1 : Draw a rough sketch of a triangle ABC and mark the given measures

(Can you mark the perimeter of triangle ?)

Analysis : Draw a line segment, say XY equal to perimeter of ΔABC i.e., AB + BC + CA. Make angles $\angle YXL$ equal to $\angle B$ and $\angle XYM$ equal to $\angle C$ and bisect them.

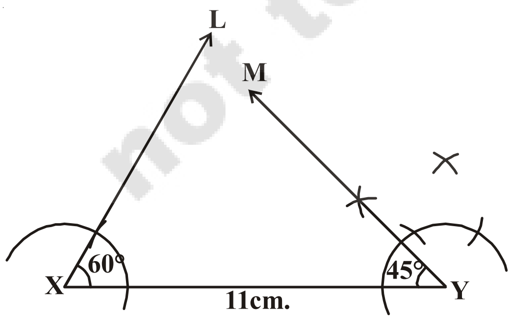
Let these bisectors intersect at a point A.

Draw perpendicular bisectors of AX to intersect XY at B and the perpendicular bisector of AY to intersect it at C. Then by joining AB and AC, we get required triangle ABC.



Step 2: Draw a line segment XY = 11 cm

(As XY = AB + BC + CA)



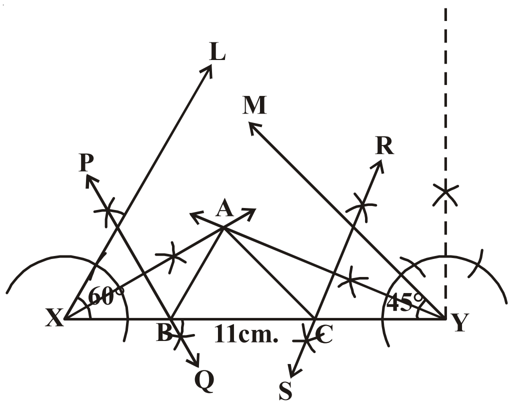
Step 3 : Construct $\angle YXL = 60^o$ and $\angle XYM = 45^o$ and draw bisectors of these angles.

Step 4 : Let the bisectors of these angles intersect at a point A and join AX or AY.

Step 5 : Draw perpendicular bisectors of AX and AY to intersect $\overrightarrow{XY}$ at B and C respectively

Join AB and AC.

Then, ABC is the required triangle.



You can justify the construction as follows

Proof: B lies on the perpendicular bisector PQ of AX

$\therefore$ XB = AB and similarly CY = AC

This gives AB + BC + CA = XB + BC + CY

= XY

Again $\angle BAX = \angle AXB$ ($\because$ XB = AB in ΔAXB) and

$\angle ABC = \angle BAX + \angle AXB$

(Exterior angle of ΔABC).

$= 2\angle AXB$

$= \angle YXL$

$= 60^o$.

Similarly $\angle ACB = \angle XYM = 45^o$ as required

$\therefore$ $\angle B = 60°$ and $\angle C = 45°$ as given are constructed.

TRY THESE

Can you draw the triangle with the same measurements in alternate way?

(**Hint**: Take $\angle YXL = \frac{60°}{2} = 30°$

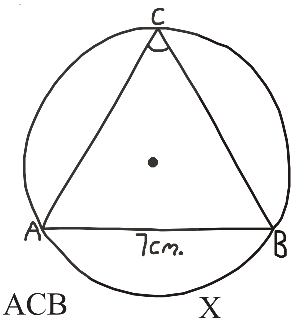
and $\angle XYM = \frac{45°}{2} = 22\frac{1}{2}°$)

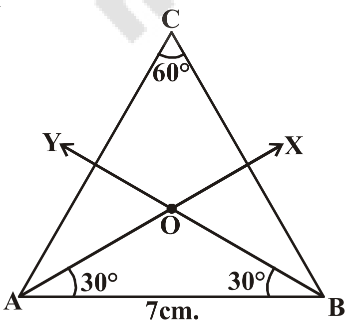
13.3.4 Construction : To construct a circle segment given a chord and a given angle.

Example-8. Construct a segment of a circle on a chord of length 7cm. and containing an angle of 60°.

Solution : Steps of construction.

Step-1: Draw a rough sketch of a circle and a segment contains an angle 60°. (Draw major segment Why?) Can you draw a circle without a centre?





Analysis: Let 'O' be the centre of the circle. Let AB be the given chord and ACB be the required segment of the circle containing an angle C = 60°.

Let $\overset{\frown}{AXB}$ be the arc subtending the angle at C.

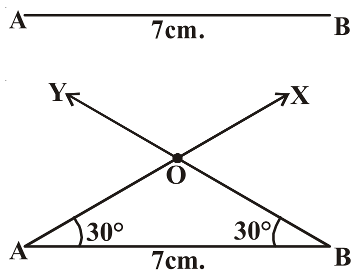
Since $\angle ACB = 60°$, $\angle AOB = 60° \times 2 = 120°$

In ΔOAB, OA=OB (radii of same circle)

$\therefore$ $\angle OAB = \angle OBA = \frac{180° - 120°}{2} = \frac{60°}{2} = 30°$

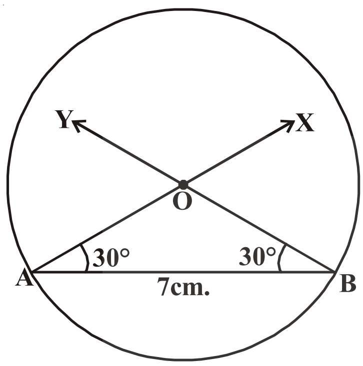
So we can draw ΔOAB then draw a circle with radius equal to OA or OB.

Step-2 : Draw a line segment AB = 7cm.



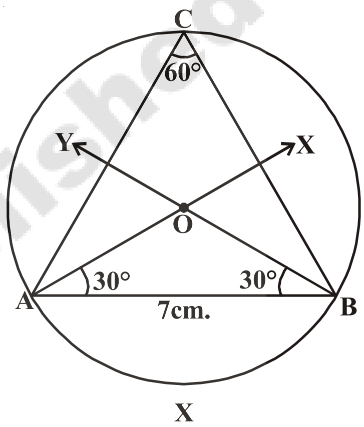
Step-3 : Draw $\overrightarrow{AX}$ such that ∠BAX = 30° and draw $\overrightarrow{BY}$ such that ∠YBA = 30° to intersect $\overrightarrow{AX}$ at O .

[Hint : Construct 30° angle by bisecting 60° angle]



Step-4 : With centre 'O' and radius OA or OB, draw the circle.

Step-5 : Mark a point 'C' on the arc of the circle. Join AC and BC. We get ∠ACB = 60°



Thus ACB is the required circle segment.

Let us justify the construction

Proof : OA = OB (radii of circle).

∴ ∠OAB + ∠OBA = 30° + 30° = 60°

∴ ∠AOB = 180° - 60° = 120°

AXB Subtends an angle of 120° at the centre of the circle.

∴ $\angle ACB = \frac{120°}{2} = 60°$

∴ ACB is the required segment of a circle.

TRY THESE

What happen if the angle in the circle segment is right angle? What kind of segment do you obtain? Draw the figure and give reason.

EXERCISE - 13.2

1. Construct ΔABC in which BC = 7 cm, ∠B = 75° and AB + AC = 12 cm.
2. Construct ΔPQR in which QR = 8 cm, ∠B = 60° and AB – AC = 3.5 cm
3. Construct Δ XYZ in which ∠Y = 30°, ∠Z = 60° and XY + YZ + ZX = 10 cm.

4. Construct a right triangle whose base is 7.5cm. and sum of its hypotenuse and other side is 15cm.
5. Construct a segment of a circle on a chord of length 5cm. containing the following angles.

   i. 90° ii. 45° iii. 120°

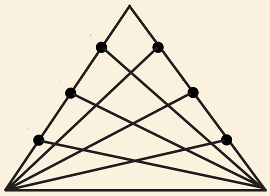
WHAT WE HAVE DISCUSSED?

1. A geometrical construction is the process of drawing geometrical figures using only two instruments - an ungraduated ruler and a compass.
2. Construction of geometrical figures of the following with justifications (Logical proofs)
   - Perpendicular bisector of a given line segment.
   - bisector of a given angle.
   - Construction of 60° angle at the initial point of a given ray.
3. To construct a triangle, given its base, a base angle and the sum of other two sides.
4. To construct a triangle given its base, a base angle and the difference of the other two sides.
5. To construct a triangle, given its perimeter and its two base angle.
6. To construct a circle segment given a chord and an angle.

Brain Teaser

How many triangles are threre in the figure ?

(It is a 'Cevian' write formula of a triangle - named in honour of Mathematician Ceva)



(**Hint :** Let the number of lines drawn from each vertex to the opposite side be 'n')