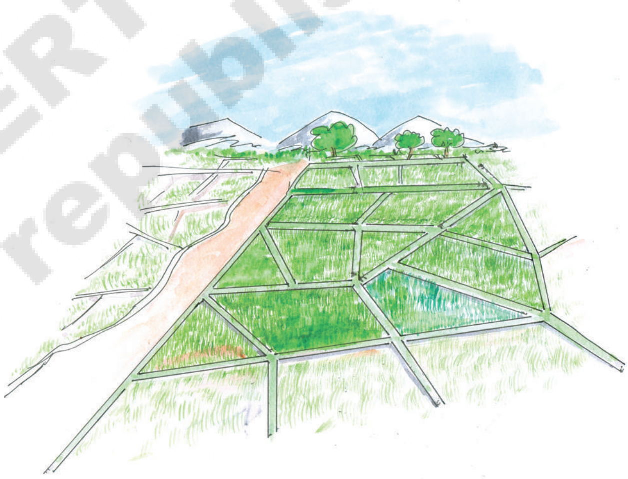
11 Areas

11.1 Introduction

Have you seen agricultural fields around your village or town? The land is divided amongst various farmers and there are many fields. Are all the fields of the same shape and same size? Do they have the same area? If a field has to be further divided among some persons, how will they divide it? If they want equal area, what can they do?

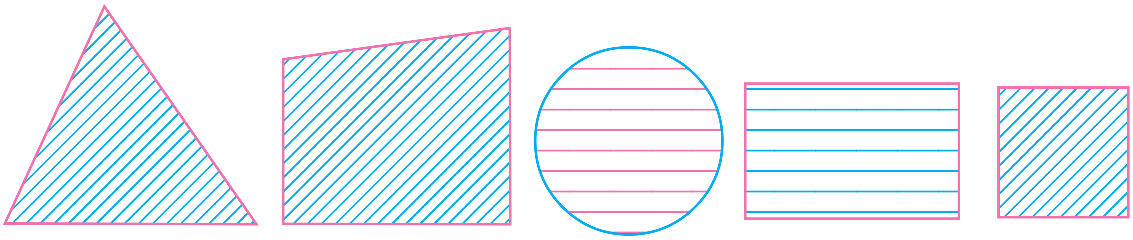
How does a farmer estimate the amount of fertilizer or seed needed for field? Does the size of the field have anything to do with this number?



The earliest and the most important reason for the initiation of the study of geometry is agricultural organisation. This includes measuring the land, dividing it into appropriated parts and recasting boundaries of the fields for the sake of convenience. In history you may have discussed the floods of river Nile (Egypt) and the land markings generated later . Some of these fields resemble the basic shapes such as square, rectangle trapezium, parallelograms etc., and some are in irregular shapes. For these basic shapes, we work out rules that would give the areas using a few lengths and measurements. We would study some of these in this chapter. We will learn how to calculate areas of triangles, squares, rectangles and quadrilaterals by using some formulae. We will also explore the basis of those formulae. We will discuss how are they derived ? What do we mean by 'area'?

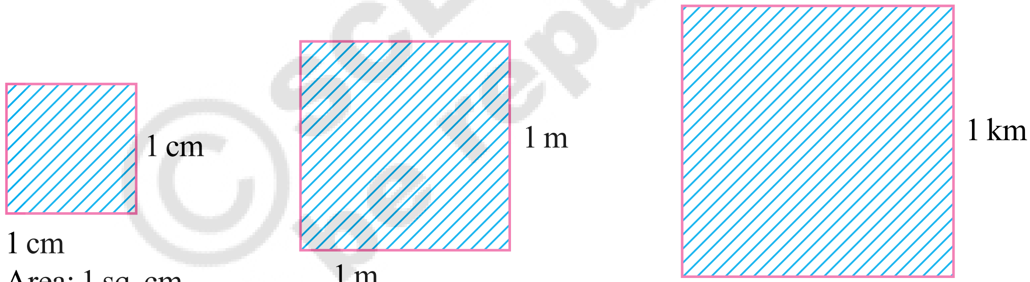
11.2 Area of Planar regions

You may recall that the part of the plane enclosed by a simple closed figure is called a planar region corresponding to that figure. The magnitude or measure of this planar region is its area.



A planar region consists of a boundary and an interior region. How can we measure the area of this? The magnitude of measure of these regions (i.e. areas) is always expressed with a positive real number (in some unit of area) such as 10 cm^2, 215 m^2, 2 km^2, 3 hectares etc. So, we can say that area of a figure is a number (in some unit of area) associated with the part of the plane enclosed by the figure.

The unit area is the area of a square of a side of unit length. Hence **square centimeter** (or 1cm^2) is the area of a square drawn on a side one centimeter in length.



Area: 1 sq. cm

Area : 1 sq. m

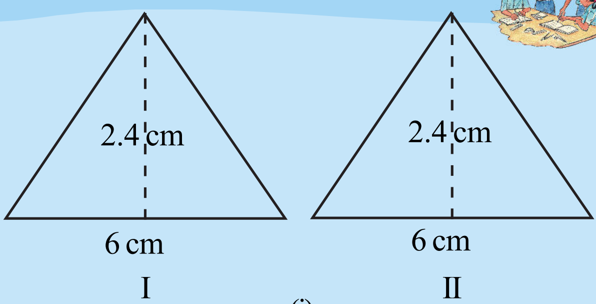
Area : 1 sq. km

The terms square meter (1m^2), square kilometer (1km^2), square millimeter (1mm^2) are to be understood in the same sense. We are familiar with the concept of congruent figures from earlier classes. Two figures are congruent if they have the same shape and the same size.

ACTIVITY

Observe Figure I and II. Find the area of both. Are the areas equal?

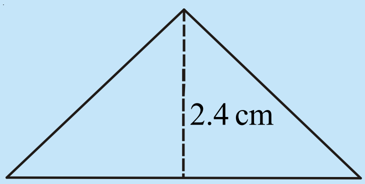
Trace these figures on a sheet of paper, cut them. Cover fig. I with fig. II. Do they cover each other completely? Are they congruent?



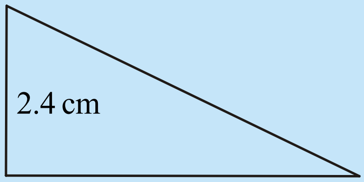
(i)

Observe fig. III and IV. Find the areas of both. What do you notice?

Are they congruent?

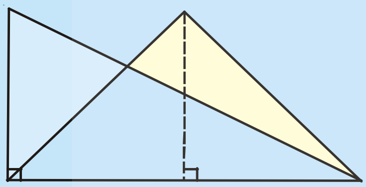


5 cm
III



5 cm
IV

Now, trace these figures on sheet of paper. Cut them let us cover fig. III by fig. IV by conciding their bases (length of same side).

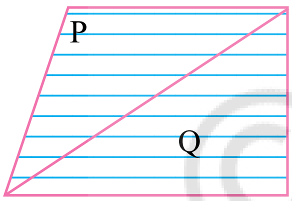


V

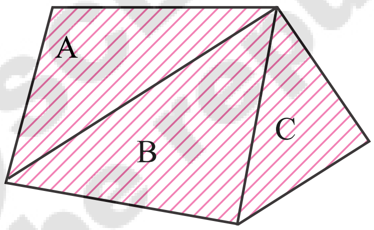
As shown in figure V are they covered completely?

We conclude that Figures I and II are congruent and equal in area. But figures III and IV are equal in area but they are not congruent.

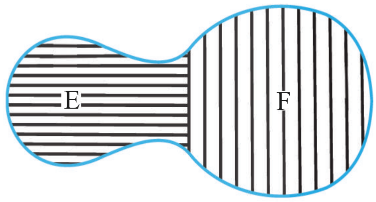
Now consider the figures given below:



X



Y



Z

You may observe that planar region of figures X, Y, Z is made up of two or more planar regions. We can easily see that

Area of figure X = Area of figure P + Area of figure Q.

Similarly area of (Y) = area of (A) + area of (B) + area of (C)

area of (Z) = area of (E) + area of (F).

Thus the area of a figure is a number (in some units) associated with the part of the plane enclosed by the figure with the following properties.

(**Note :** We use area of a figure (X) briefly as ar(X) from now onwards)

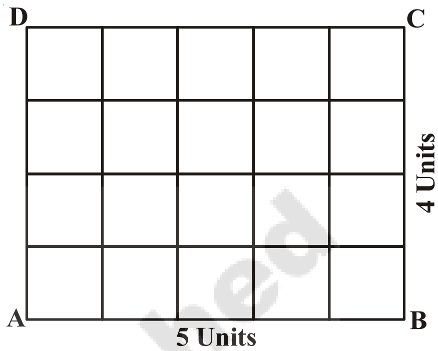
(i) The areas of two congruent figures are equal.

If A and B are two congruent figures, then ar(A) = ar(B)

(ii) The area of a figure is equal to the sum of the areas of finite number of parts of it. If a planar region formed by a figure X is made up of two non-overlapping planar regions formed by figures P and Q then ar(X) = ar(P) + ar(Q).

11.3 Area of Rectangle

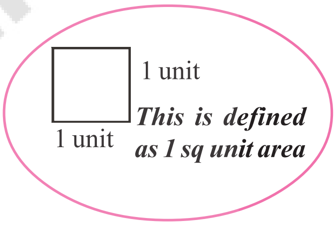
If the number of units in the length of a rectangle is multiplied by the number of units in its breadth, the product gives the number of square units in the area of rectangle



Let ABCD represent a rectangle whose length $\overline{AB}$ is 5 units and breadth $\overline{BC}$ is 4 units.

Divide $\overline{AB}$ into 5 equal parts and $\overline{BC}$ into 4 equal parts and through the points of division of each line draw parallels to the other. Each compartment in the rectangle represents one square unit (why ?)

$\therefore$ The rectangle contains (5 units $\times$ 4 units). That is 20 square units.



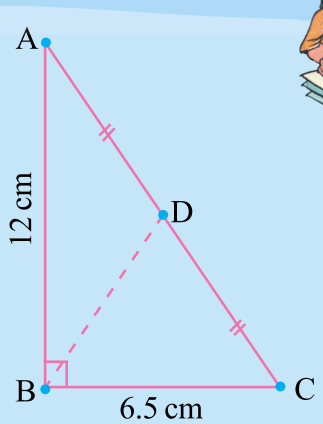
Similarly, if the length is 'a' units and breadth is 'b' units then the area of rectangle is 'ab' square units. That is "length $\times$ breadth" square units gives the area of a rectangle.

Think, Discuss and Write

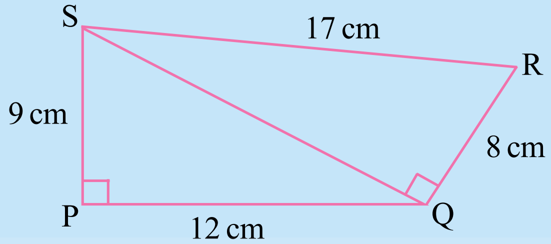
1. If 1cm represents 5m, what would be an area of 6 square cm. represent ?
2. Rajni says 1 sq.m = 100^2 sq.cm. Do you agree? Explain.

Exercise - 11.1

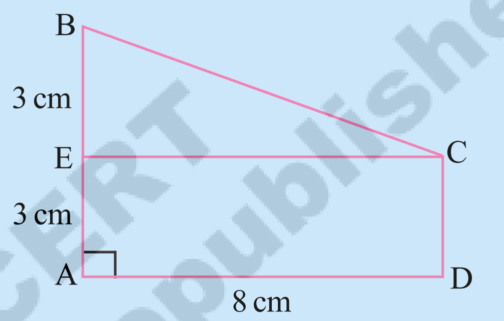
1. In ΔABC, $\angle ABC = 90^\circ$, AD = DC, AB = 12 cm and BC = 6.5 cm. Find the area of ΔADB.



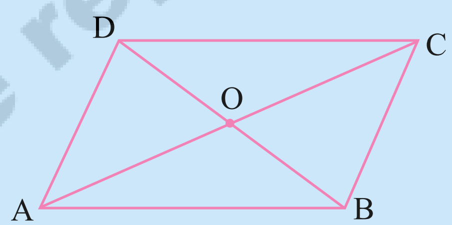
2. Find the area of a quadrilateral PQRS in which ∠QPS = ∠SQR = 90°, PQ = 12 cm, PS = 9 cm, QR = 8 cm and SR = 17 cm (**Hint:** PQRS has two parts)



3. Find the area of trapezium ABCD as given in the figure in which ADCE is a rectangle. (**Hint:** ABCD has two parts)



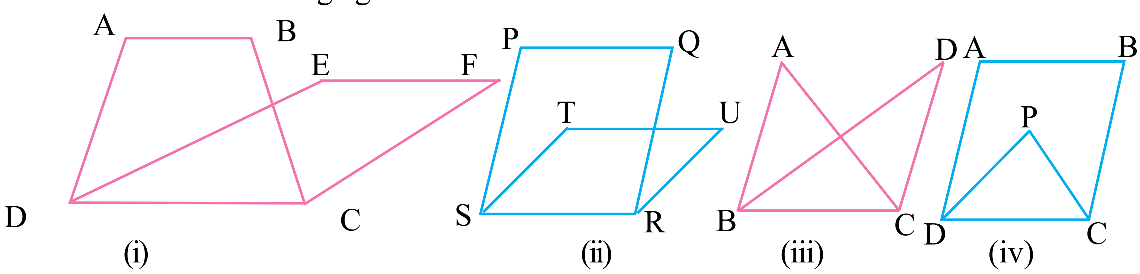
4. ABCD is a parallelogram. The diagonals AC and BD intersect each other at 'O'. Prove that ar(ΔAOD) = ar(ΔBOC). (**Hint:** Congruent figures have equal area)



11.4 Figures on the same base and between the same parallels

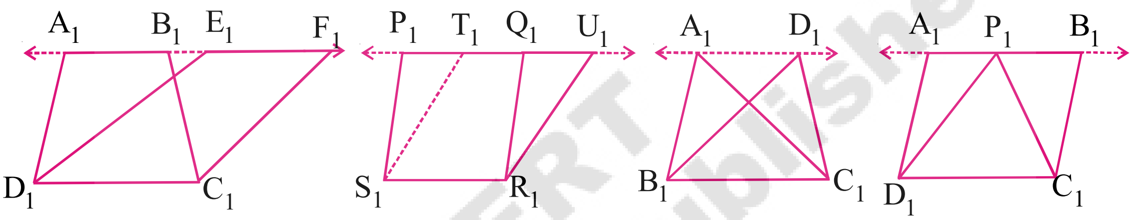
We shall now study some relationships between the areas of some geometrical figures under the condition that they lie on the same base and between the same parallels. This study will also be useful in understanding of some results on similarity of triangles.

Look at the following figures.



In Fig(i) a trapezium ABCD and parallelogram EFCD have a common side CD. We say that trapezium ABCD and parallelogram EFCD are on the same base CD. Similarly in fig(ii) the base of parallelogram PQRS and parallelogram TURS is the same. In fig(iii) Triangles ABC and DBC have the same base BC. In Fig(iv) parallelogram ABCD and triangle PCD lie on DC so, all these figures are of geometrical shapes are therefore on the same base. They are however not between the same parallels as AB does not overlap EF and PQ does not overlap TU etc. Neither the points A, B, E, F are collinear nor the points P, Q, T, U. What can you say about Fig(iii) and Fig (iv)?

Now observe the following figures.



(v) (vi) (vii) (viii)

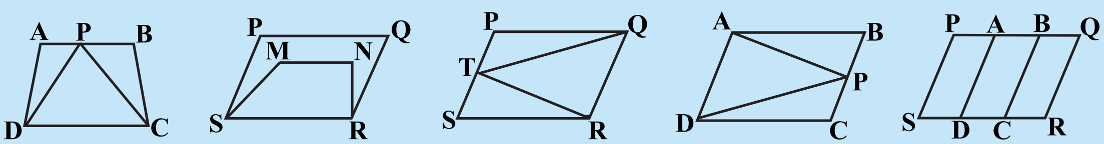
What difference have you observed among the figures? In Fig(v), We say that trapezium $A_1B_1C_1D_1$ and parallelogram $E_1F_1C_1D_1$ are on the same base and between the same parallels A_1F_1 and D_1C_1. The points A_1, B_1, E_1, F_1 are collinear and $A_1F_1 \parallel D_1C_1$. Similarly in fig. (vi) parallelograms $P_1Q_1R_1S_1$ and $T_1U_1R_1S_1$ are on the same base S_1R_1 and between the same parallels P_1U_1 and S_1R_1. Name the other figures on the same base and the parallels between which they lie in fig. (vii) and (viii).

So, two figures are said to be on the same base and between the same parallels, if they have a common base (side) and the vertices (or the vertex) opposite to the common base of each figure lie on a line parallel to the base.

THINK, DISCUSS AND WRITE

Which of the following figures lie on the same base and between the same parallels?

In such a cases, write the common base and the two parallels.



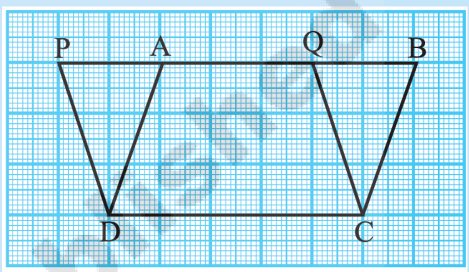
(a) (b) (c) (d) (e)

11.5 PARALLELOGRAMS ON THE SAME BASE AND BETWEEN THE SAME PARALLELS

Now let us try to find a relation, if any, between the areas of two parallelograms on the same base and between the same parallels. For this, let us perform the following activity.

ACTIVITY

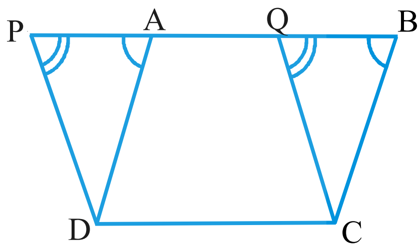
Take a graph sheet and draw two parallelograms ABCD and PQCD on it as show in the Figure-



The parallelograms are on the same base DC and between the same parallels PB and DC Clearly the part DCQA is common between the two parallelograms. So if we can show that ΔDAP and ΔCBQ have the same area then we can say ar(PQCD) = ar(ABCD).

Theorem-11.1 : Parallelograms on the same base and between the same parallels are equal in area.

Proof: Let ABCD and PQCD are two parallelograms on the samebase DC and between the parallel lines DC and PB.



In ΔDAP and ΔCBQ

PD || CQ and PB is transversal $\angle DPA = \angle CQB$

and AD || CB and PB is transversal $\angle DAP = \angle CBQ$

also PD = QC as PQCD is a parallelogram.

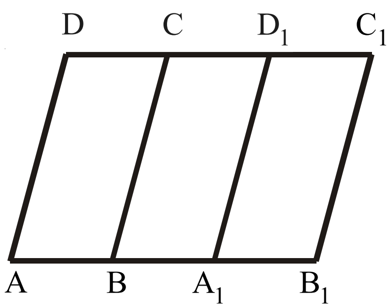
Hence ΔDAP and ΔCBQ are congruent and have equal areas.

So we can say ar(PQCD) = ar (AQCD) + ar(DAP)

= ar(AQCD) + ar(CBQ) = ar(ABCD)

You can verify by counting the squares of these parallelogram as drawn in the graph sheet.

Can you explain how to count full squares below half a square, above half a square on graph sheet.



Reshma argues that the parallelograms between same parallels need not have a common base for equal area. They only need to have an equal base. To understand her statement look at the adjacent figure.

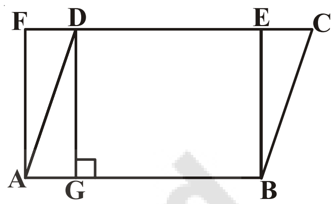
If $AB = A_1B_1$ When we cut out parallelogram $A_1B_1C_1D_1$ and place it over parallelogram ABCD, A would concide in with A_1 and B with B_1 and $\overline{C_1D_1}$ coincide with CD. Thus these are equal in area. Thus the parallelogram with the equal base can be considered to be on the same base for the purposes of studying their geometrical properties.

Let us now take some examples to illustrate the use of the above Theorem.

Example-1. ABCD is parallelogram and ABEF is a rectangle and DG is perpendicular on AB.

Prove that (i) ar (ABCD) = ar(ABEF)

(ii) ar (ABCD) = AB × DG



Solution : (i) A rectangle is also a parallelogram

$\therefore$ ar(ABCD) = ar(ABEF) ..... (1)

(Parallelograms lie on the same base and between the same parallels)

(ii) ar(ABCD) = ar(ABEF) ($\because$ from (1))

= AB × BE ($\because$ ABEF is a rectangle)

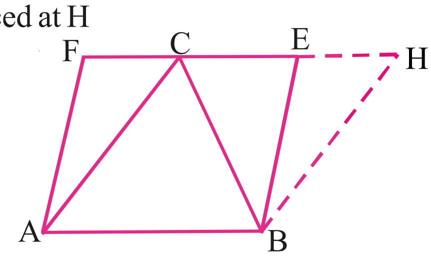
= AB × DG ($\because$ DG $\perp$ AB and DG = BE)

Therefore ar(ABCD) = AB × DG

From the result, we can say that "area of a parallelogram is the product of its any side and the corresponding altitude".

Example-2. Triangle ABC and parallelogram ABEF are on the same base, AB as in between the same parallels AB and EF. Prove that ar(ΔABC) = $\frac{1}{2}$ ar(|| gm ABEF)

Solution : Through B draw BH || AC to meet FE produced at H



$\therefore$ ABHC is a parallelogram

Diagonal BC divides it into two congruent triangles

$\therefore$ ar(ΔABC) = ar(ΔBCH)

= $\frac{1}{2}$ ar (|| gm ABHC)

But || gm ABHC and || gm ABEF are on the same base AB and between same parallels AB and EF

$\therefore$ ar(|| gm ABHC) = ar(|| gm ABEF)

Hence ar(ΔABC) = $\frac{1}{2}$ ar (|| gm ABEF)

From the result, we say that "the area of a triangle is equal to half the area of the parallelogram on the same base and between the same parallels".

Example-3. Find the area of a figure formed by joining the mid-points of the adjacent sides of a rhombus with diagonals 12 cm. and 16 cm.

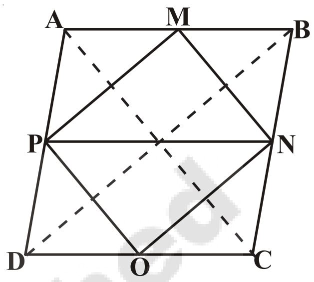
Solution : Join the mid points of AB, BC, CD, DA of a rhombus ABCD and name them M, N, O and P respectively to form a figure MNOP.

What is the shape of MNOP thus formed? Give reasons?

Join the line PN, then PN || AB and PN || DC (How?)

We know that if a triangle and a parallelogram are on the same base and between the same parallels, the area of the triangle is equal to one-half area of the parallelogram.

From the above result parallelogram ABNP and triangle MNP are on the same base PN and in between same parallel lines PN and AB.



$\therefore$ ar $\Delta MNP = \frac{1}{2}$ ar ABPN .....(i)

Similarly ar $\Delta PON = \frac{1}{2}$ ar PNCD .....(ii)

and Area of rhombus $= \frac{1}{2} \times d_1 d_2$

From (1), (ii) and (iii) we get

$$ar(MNOP) = ar(\Delta MNP) + ar(\Delta PON)$$

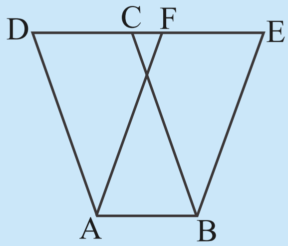
$$= \frac{1}{2} ar(ABNP) + \frac{1}{2} ar(ABCD)$$

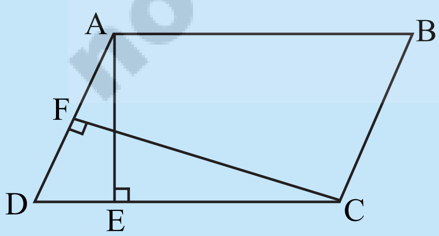
$$= \frac{1}{2} ar(\text{rhombus } ABCD)$$

$$= \frac{1}{2}\left(\frac{1}{2} \times 12 \times 16\right) = 48 \text{ cm.}^2$$

Exercise - 11.2

1. The area of parallelogram ABCD is 36 cm^2. Calculate the height of parallelogram ABEF if AB = 4.2 cm.

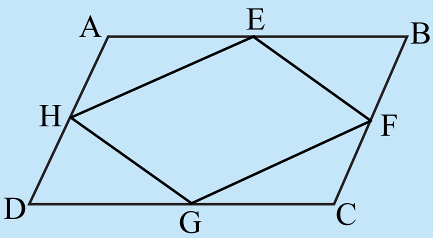




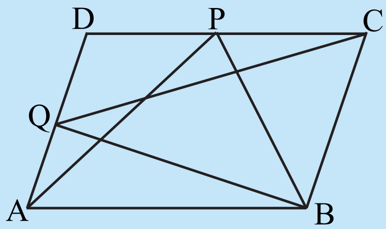
2. ABCD is a parallelogram. AE is perpendicular on DC and CF is perpendicular on AD.

   If AB = 10 cm, AE = 8 cm and CF = 12 cm. Find AD.

3. If E, F G and H are respectively the midpoints of the sides AB, BC, CD and AD of a parallelogram ABCD, show that $ar(EFGH) = \frac{1}{2} ar(ABCD)$.

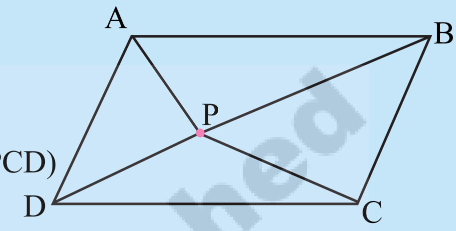


4. What figure do you get, if you join ΔAPM, ΔDPO, ΔOCN and ΔMNB in the example 3.



5. P and Q are any two points lying on the sides DC and AD respectively of a parallelogram ABCD show that ar(ΔAPB) = ar Δ(BQC).

6. P is a point in the interior of a parallelogram ABCD. Show that

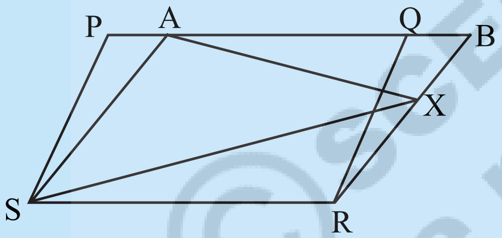


(i) $ar(\Delta APB) + ar(\Delta PCD) = \frac{1}{2} ar(ABCD)$

(ii) $ar(\Delta APD) + ar(\Delta PBC) = ar(\Delta APB) + ar(\Delta PCD)$

(Hint : Through P, draw a line parallel to AB)

7. Prove that the area of a trapezium is half the sum of the parallel sides multiplied by the distance between them.

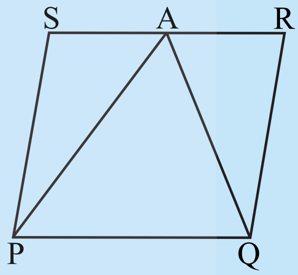


8. PQRS and ABRS are parallelograms and X is any point on the side BR. Show that

(i) ar(PQRS) = ar(ABRS)

(ii) $ar(\Delta AXS) = \frac{1}{2} ar(PQRS)$

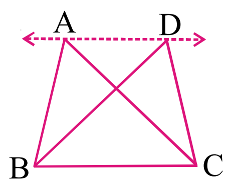
9. A farmer has a field in the form of a parallelogram PQRS as shown in the figure. He took the mid- point A on RS and joined it to points P and Q. In how many parts of field is divided? What are the shapes of these parts ?



The farmer wants to sow groundnuts which are equal to the sum of pulses and paddy. How should he sow? State reasons?

10. Prove that the area of a rhombus is equal to half of the product of the diagonals.

11.6 Triangles on the same base and between the same parallels



We are looking at figures that lie on the same base and between the same parallels. Let us have two triangles ABC and DBC on the same base BC and between the same parallels, AD and BC.

What can we say about the areas of such triangles? Clearly there can be infinite number of ways in which such pairs of triangle between the same parallels and on the same base can be drawn.

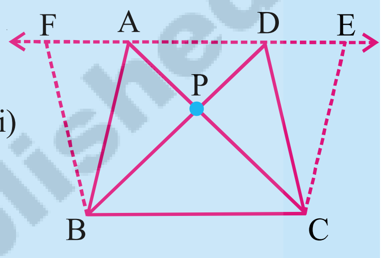
Let us perform an activity.

ACTIVITY

Draw pairs of triangles one the same base or (equal bases) and between the same parallels on the graph sheet as shown in the Figure.

Let ΔABC and ΔDBC be the two triangles lying on the same base BC and between parallels BC and FE.

Draw CE || AB and BF || CD. Parallelograms AECB and FDCB are on the same base BC and are between the same parallels BC and EF.



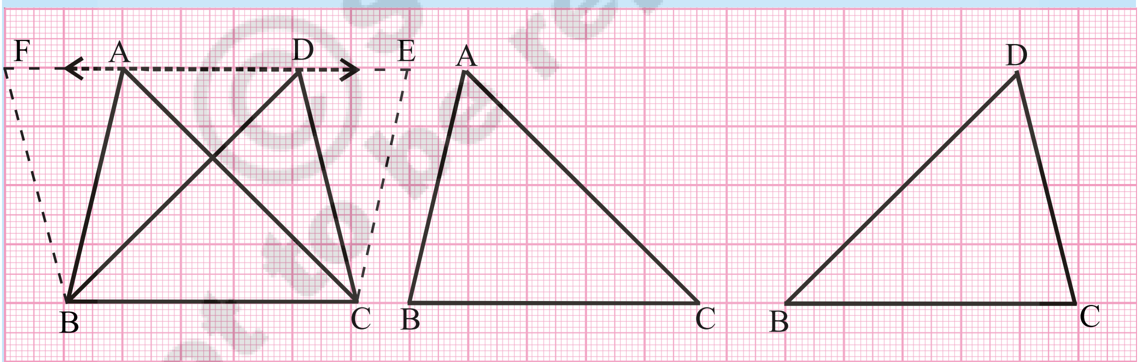
Thus ar(AECB) = ar(FDCB). (How ?)

We can see $ar(\Delta ABC) = \frac{1}{2}$ ar(Parallelogram AECB) ...(i)

and $ar(\Delta DBC) = \frac{1}{2}$ ar(Parallelogram FDCB) .... (ii)(

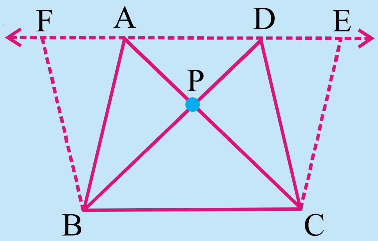
From (i) and (ii), we get $ar(\Delta ABC) = ar(\Delta DBC)$.

You can also find the areas of ΔABC and ΔDBC by the method of counting the squares in graph sheet as we have done in the earlier activity and check the areas are whether same.



THINK, DISCUSS AND WRITE

Draw two triangles ABC and DBC on the same base and between the same parallels as shown in the figure with P as the point of intersection of AC and BD. Draw CE || BA and BF || CD such that E and F lie on line AD.



Can you show $ar(\Delta PAB) = ar(\Delta PDC)$

Hint : These triangles are not congruent but have equal areas.

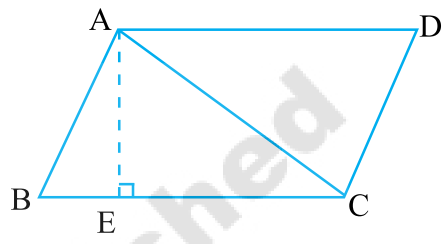
Corollary-1 : Show that the area of a triangle is half the product of its base (or any side) and the corresponding attitude (height).

Proof : Let ABC be a triangle. Draw AD || BC such that CD = BA.

Now ABCD is a parallelogram one of whose diagonals is AC.

We know $\Delta ABC \cong \Delta ACD$

So arΔABC = arΔACD (Congruent triangles have equal area)



Therefore, $ar\Delta ABC = \frac{1}{2} ar(ABCD)$

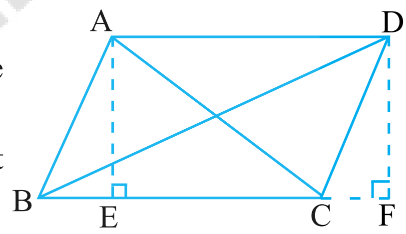
Draw $AE \perp BC$

We know $ar(ABCD) = BC \times AE$

We have $ar(\Delta ABC) = \frac{1}{2} ar(ABCD) = \frac{1}{2} \times BC \times AE$

So $ar\Delta ABC = \frac{1}{2} \times$ base BC $\times$ Corresponding attitude AE.

Theorem-11.2 : Two triangles having the same base (or equal bases) and equal areas will lie between the same parallels.



Observe the figure. Name the triangles lying on the same base BC. What are the heights of ΔABC and ΔDBC?

If two triangles have equal area and equal base, what will be their heights? Are A and D collinear?

Let us now take some examples to illustrate the use of the above results.

Example 4. Show that the median of a triangle divides it into two triangles of equal areas.

Solution : Let ABC be a triangle and Let AD be one of its medians.

In ΔABD and ΔADC the vertex is common and these bases BD and DC are equal.

Draw $AE \perp BC$.

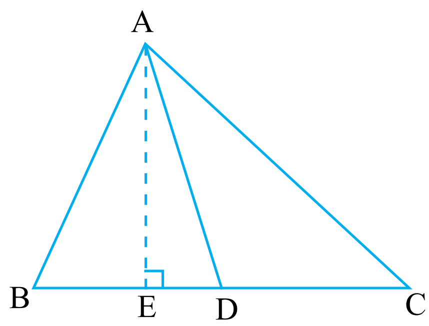
$$\text{Now ar } (\Delta ABD) = \frac{1}{2} \times \text{base BD} \times \text{altitude of } \Delta ADB$$

$$= \frac{1}{2} \times BD \times AE$$

$$= \frac{1}{2} \times DC \times AE \quad (\because BD = DC)$$

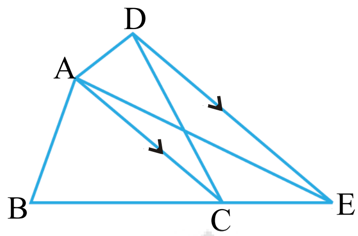
$$= \frac{1}{2} \times \text{base DC} \times \text{altitude of } \Delta ACD$$

$$= \text{ar } \Delta ACD$$



Hence ar (ΔABD) = ar (ΔACD)

Example-5. In the figure, ABCD is a quadrilateral. AC is the diagonal and DE || AC and also DE meets BC produced at E. Show that ar(ABCD) = ar(ΔABE).



Solution : ar(ABCD) = ar(ΔABC) + ar(ΔDAC)

ΔDAC and ΔEAC lie on the same base $\overline{AC}$

and between the parallels DE || AC

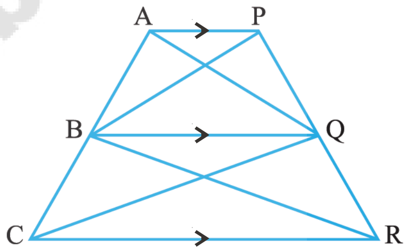
ar(ΔDAC) = ar(ΔEAC) (Why?)

Adding areas of same figures on both sides.

ar(ΔDAC) + ar(ΔABC) = ar(ΔEAC) + ar(ΔABC)

Hence ar(ABCD) = ar(ΔABE)

Example 6. In the figure , AP || BQ || CR. Prove that ar(ΔAQC) = ar(ΔPBR).



Solution : ΔABQ and Δ PBQ lie on the same base BQ and between the same parallels AP || BQ.

ar(ΔABQ) = ar(ΔPBQ) ...(1)

Similarly

ar (ΔCQB) = ar (ΔRQB) (same base BQ and BQ || CR) ...(2)

Adding results (1) and (2)

ar(ΔABQ) + ar(ΔCQB) = ar(ΔPBQ) + ar(ΔRQB)

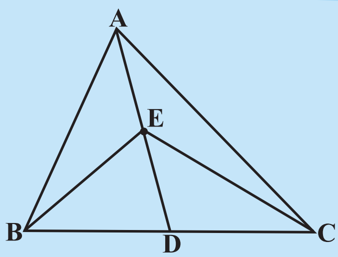
Hence ar ΔAQC = ar ΔPBR

Exercise - 11.3

1. In a triangle ABC (see figure), E is the midpoint of median AD, show that

   (i) ar ΔABE = ar ΔACE

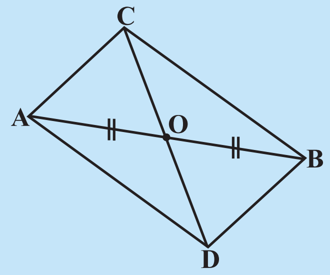
   (ii) arΔABE $= \frac{1}{4}$ar(ΔABC)

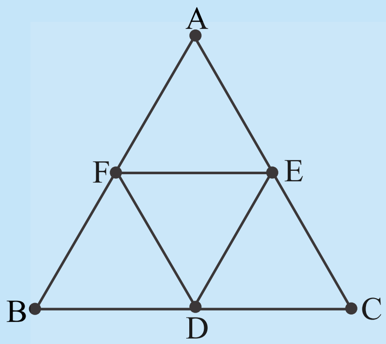


2. Show that the diagonals of a parallelogram divide it into four triangles of equal area.

3. In the figure, ΔABC and ΔABD are two triangles on the same base AB. If line segment CD is bisected by $\overline{AB}$ at O, show that

   ar (ΔABC) = ar (ΔABD).





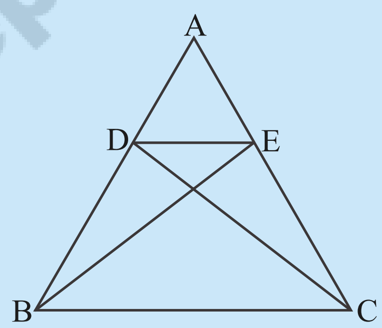
4. In the figure, ΔABC, D, E, F are the midpoints of sides BC, CA and AB respectively. Show that

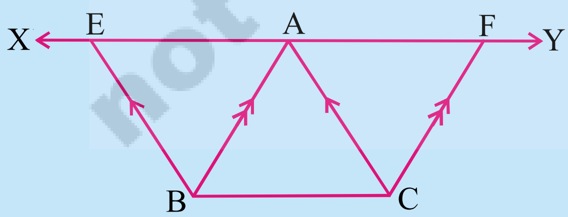
   (i) BDEF is a parallelogram

   (ii) $ar(\Delta DEF) = \frac{1}{4} ar(\Delta ABC)$

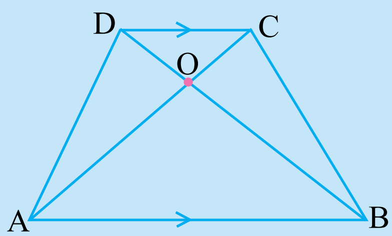
   (iii) $ar(BDEF) = \frac{1}{2} ar(\Delta ABC)$

5. In the figure D, E are points on the sides AB and AC respectively of ΔABC such that $ar(\Delta DBC) = ar(\Delta EBC)$. Prove that $DE \parallel BC$.



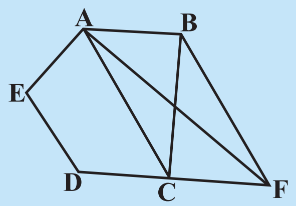


6. In the figure, XY is a line parallel to BC is drawn through A. If $BE \parallel CA$ and $CF \parallel BA$ are drawn to meet XY at E and F respectively. Show that $ar(\Delta ABE) = ar(\Delta ACF)$.



7. In the figure, diagonals AC and BD of a trapezium ABCD with $AB \parallel DC$ intersect each other at O. Prove that

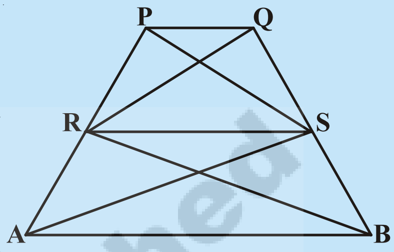
   $ar(\Delta AOD) = ar(\Delta BOC)$.



8. In the figure, ABCDE is a pentagon. A line through B parallel to AC meets DC produced at F. Show that

   (i) ar (ΔACB) = ar (ΔACF)

   (ii) ar (AEDF) = ar (ABCDE)



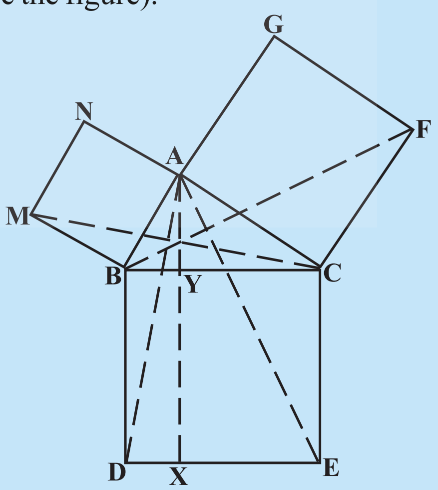
9. In the figure, if ar ΔRAS = ar ΔRBS and [ar (ΔQRB) = ar(ΔPAS) then show that both the quadrilaterals PQSR and RSBA are trapeziums.

10. A villager Ramayya has a plot of land in the shape of a quadrilateral. The grampanchayat of the village decided to take over some portion of his plot from one of the corners to construct a school. Ramayya agrees to the above proposal with the condition that he should be given equal amount of land in exchange of his land adjoining his plot so as to form a triangular plot. Explain how this proposal will be implemented. (Draw a rough sketch of plot).

Think, Discuss and Write

ABC is a right triangle right angled at A. BCED, ACFG and ABMN are squares on the sides BC, CA and AB respectively. Line segments AX ⊥ DE meets BC at Y. and DE at X. Join AD, AE also BF and CM (See the figure).

Show that



(i) ΔMBC ≅ ΔABD

(ii) ar(BYXD) = 2ar (ΔMBC)

(iii) ar(BYXD) = ar(ABMN)

(iv) ΔFCB ≅ ΔACE

(v) ar(CYXE) = 2 ar(FCB)

(vi) ar(CYXE) = ar (ACFG)

(vii) ar(BCED) = ar(ABMN) + ar(ACFG)

Can you write the result (vii) in words ? This is a famous theorem of Pythagoras. You shall learn a simpler proof in this theorem in class X.

WHAT WE HAVE DISCUSSED

In this chapter we have discussed the following.

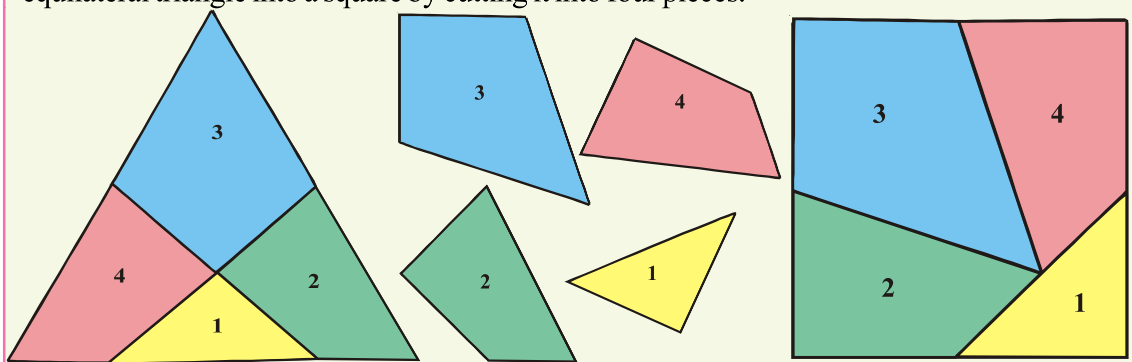
1. Area of a figure is a number (in some unit) associated with the part of the plane enclosed by that figure.
2. Two congruent figures have equal areas but the converse need not be true.
3. If X is a planer region formed by two non-overlapping planer regions of figures P and Q, then ar(X) = ar(P) + ar(Q)
4. Two figures are said to be on the same base and between the same parallels, if they have a common base (side) and the vertices (on the vertex) opposite to the common base of each figure lie on a line parallel to the base.
5. Parallelograms on the same base (or equal bases) and between the same parallels are equal in area.
6. Area of a parallelogram is the product of its base and the corresponding altitude.
7. Parallelogram on the same base (or equal bases) and having equal areas lie between the same parallels.
8. If a parallelogram and a triangle are on the same base and between the same parallels, then area of the triangle is half the area of the parallelogram.
9. Triangles on the same base (or equal bases) and between the same parallels are equal in area.
10. Triangles on the same base (or equal bases) and having equal areas lie between the same parallels.

DO YOU KNOW?

A PUZZLE (AREAS)

German mathematician David Hilbert (1862-1943) first proved that any polygon can be transformed into any other polygon of equal area by cutting it into a finite number of pieces.

Let us see how an English puzzlist, Henry Ernest Dudency (1847 - 1930) transforms an equilateral triangle into a square by cutting it into four pieces.



Try to make some more puzzles using his ideas and enjoy.